DARBOUX TRANSFORMS AND SPECTRAL CURVES OF CONSTANT MEAN CURVATURE SURFACES REVISITED

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1. Introduction

Surfaces of non–zero constant mean curvature in Euclidean 3–space are studied from a variety of different view points. These surfaces are critical with respect to the variation of area with constrained volume and their Euler–Lagrange equation is an important example of a geometric non–linear elliptic partial differential equation [27, 15, 19, 10]. Constant mean curvature surfaces also have a deep connection with the theory of integrable systems since the Gauss–Codazzi equation is a well known soliton equation, namely the sinh–Gordon equation. This has led to a complete classification of constant mean curvature tori in terms of periodic linear flows on Jacobians of hyperelliptic algebraic curves [20, 13, 2, 8, 14].

The essential ingredient of the integrable systems approach is that a surface $f\colon M \to \mathbb{R}^3$ of constant mean curvature has an *associated* S^1*-family* of constant mean curvature surfaces f^μ, obtained by rotating the Hopf differential of f by $\mu \in S^1$. The surfaces f^μ generally develop translational and rotational periods in $\mathbb{R}^3$ if M has non–trivial topology. Extending the circle parameter, also called the *spectral parameter*, to $\mu \in \mathbb{CP}^1$ one obtains a rational family of flat $\mathbf{SL}(2,\mathbb{C})$ connections ∇^μ over the surface M with simple poles at $\mu = 0, \infty$. This family is unitary along the unit circle $\mu \in S^1$ where it describes the associated family of constant mean curvature surfaces f^μ. When $M = T^2$ is a torus, the holonomy representation H^μ of the family ∇^μ with respect to a chosen base point p on T^2 is abelian and hence has simultaneous eigenlines. These eigenlines define a hyperelliptic algebraic curve Σ_e over the μ–parameter space $\mathbb{CP}^1$, together with a holomorphic line bundle $\mathcal{E}(p)$ over Σ_e. Changing the base point $p \in T^2$ does not change this *eigenline spectral curve* Σ_e, but the *eigenline bundles* $\mathcal{E}(p)$ sweep out a 2–dimensional subtorus of the Jacobian of Σ_e.

Recently a more general notion of a spectral curve has been introduced [4, 25, 23] for any conformally immersed torus $f\colon T^2 \to S^4$. This *multiplier spectral curve* Σ_m does not rely upon the existence of a family of flat connections and it arises rather geometrically as a desingularisation of the set of all Darboux transforms of f. By a Darboux transform of f we mean

Date: January 19, 2013.

a conformally immersed torus $\hat{f}\colon T^2 \to S^4$ for which there is a 2–sphere congruence along f touching f and half–touching $\hat{f}$. Darboux transforms include the *classical* Darboux transforms for which $\hat{f}$ also touches, rather than merely half–touches, the said sphere congruence. In the classical case both surfaces f and $\hat{f}$ are isothermic. Analytically the multiplier spectral curve of a conformally immersed torus f is given by the possible holonomies, or "multipliers", of quaternionic holomorphic sections for the quaternionic holomorphic structure D induced by f on the quaternionic bundle V/L. Here L is the pull–back under f of the tautological bundle over $S^4 = \mathbb{HP}^1$ and V is the trivial $\mathbb{H}^2$–bundle. Generically there is (up to scale) exactly one quaternionic holomorphic section for a given multiplier in Σ_m and one thereby obtains a T^2-family of holomorphic line bundles, the *kernel bundles*, over the multiplier spectral curve. Again, in the case when Σ_m has finite genus, this T^2 family of holomorphic line bundles sweeps out a subtorus [5] of the Jacobian of Σ_m. The Darboux transform of f corresponding to a multiplier $h \in \Sigma_m$ is then given by $\hat{f} = \hat{\varphi}\mathbb{H}$, where $\hat{\varphi}$ is the prolongation to V of the quaternionic holomorphic section φ of V/L with holonomy h.

This paper discusses this general approach to spectral curves in the context of constant mean curvature tori in $\mathbb{R}^3$: what are the geometric properties of the Darboux transforms of a constant mean curvature torus? Are the two spectral curves the same and how do the eigenline and kernel bundles relate?

The first question we can answer even locally: given a (simply connected) constant mean curvature surface $f\colon M \to \mathbb{R}^3$ any parallel section of the flat connection ∇^μ is quaternionic holomorphic since the connections ∇^μ all induce the same quaternionic holomorphic structure D on V/L. The prolongation of any such parallel section is therefore a Darboux transform of f, which we call a *μ-Darboux transform*. Note that there is a $\mathbb{CP}^1$-worth of parallel sections to a given μ. We show that all μ–Darboux transforms of the constant mean curvature surface $f\colon M \to \mathbb{R}^3$ are again constant mean curvature surfaces, albeit in a parallel translated $\mathbb{R}^3 \subset \mathbb{R}^4$. Amongst the classical Darboux transforms of f in $\mathbb{R}^3$, those which have constant mean curvature form a 3-dimensional hyper-surface [12]. We show that this hyper-surface consists entirely of μ-Darboux transforms with $\mu \in \mathbb{R} \setminus \{0, 1\}$. For non-real μ the only other μ-Darboux transforms which are classical occur for unitary μ and give the parallel constant mean curvature surface.

The second question concerns the global existence of Darboux transforms of a constant mean curvature torus $f\colon T^2 \to \mathbb{R}^3$. Away from the points P_0, P_∞ over $\mu = 0, \infty$ a point x on the eigenline spectral curve Σ_e of f is described by a parallel section of ∇^μ with holonomy. This section therefore gives a μ-Darboux transform $\hat{f}^x\colon T^2 \to S^4$ defined on the same 2–torus and its holonomy defines a map h into the multiplier spectral curve Σ_m. This map does not extend to an isomorphism of the two curves; the eigenline curve is

algebraic and generically smooth whereas the multiplier curve is always singular and may have infinite arithmetic genus. This discrepancy is resolved by desingularising: the lift of the map h to the normalisations extends holomorphically, thereby compactifying the normalisation of the multiplier spectral curve and yielding a biholomorphism of the two desingularised curves. The μ-Darboux transforms limit to the original constant mean curvature torus at P_0 and P_∞, yielding a geometric method for recovering the original surface. A similar result has been proven in the more general context of conformally immersed tori in the 4-sphere having finite spectral genus [4]. However, we utilise the existence of a family of flat connections in the constant mean curvature case to give a considerably simpler proof. Since the kernel and eigenline bundles lift to holomorphic line bundles which agree by construction away from a discrete set of points, they are the same bundle on the identified normalisations of the two spectral curves. This implies that any Darboux transform of a constant mean curvature torus $f\colon T^2 \to \mathbb{R}^3$ which is parameterised by the multiplier curve is a μ-Darboux transform and hence also a torus of constant mean curvature in $\mathbb{R}^3$. However not all Darboux transforms or even all classical Darboux transforms of a constant mean curvature torus must again have constant mean curvature [1, 17]. The space of all Darboux transforms may, in addition to the multiplier curve, contain countably many quaternionic and complex projective spaces. There may similarly be many μ–Darboux transforms which are not parameterised by the eigenline curve, and in the appendix we study the simple example of the standard cylinder and show that these correspond to the adding of bubbletons to the original constant mean curvature surface.

2. Darboux Transformations

We model the conformal geometry of the 4–sphere by the quaternionic projective line $S^4 = \mathbb{HP}^1$ on which the group of orientation preserving Möbius transformations acts by $\mathbf{GL}(2, \mathbb{H})$. A map $f : M \to S^4$ can be considered as a line subbundle $L \subset V$ of the trivial $\mathbb{H}^2$ bundle $V = \underline{\mathbb{H}}^2$, where the fibers of L are given by $L_p = f(p)$ for $p \in M$. In other words, $L = f^*\mathcal{T}$ is the pullback of the tautological line bundle $\mathcal{T}$ over $\mathbb{HP}^1$. Identifying the tangent bundle of $\mathbb{HP}^1$ with $\mathrm{Hom}(\mathcal{T}, \underline{\mathbb{H}}^2/\mathcal{T})$ the derivative of f is given by

$$\delta = \pi d|_L \in \Omega^1(\mathrm{Hom}(L, V/L))\,, \tag{1}$$

where d is the trivial connection on V and $\pi : V \to V/L$ is the canonical projection. Throughout the paper we denote by $\mathrm{Hom}(W_1, W_2)$ the real vector space of quaternionic linear maps between quaternionic (right) vector spaces W_1 and W_2. An immersion $f\colon M \to S^4$ is conformal [6] if and only if there exist complex structures $J \in \Gamma(\mathrm{End}(V/L))$ on V/L and $\widetilde{J} \in \Gamma(\mathrm{End}(L))$ on L such that

$$* \delta = J\delta = \delta\widetilde{J}\,, \tag{2}$$

where $*$ is the conformal structure on T^*M.

An oriented round 2–sphere in $S^4 = \mathbb{HP}^1$ is described by a complex structure $S \in \mathrm{End}(\mathbb{H}^2)$, $S^2 = -1$: points on the sphere are the fixed lines of S. In particular, the corresponding line subbundle $L_S \subset V$ of the embedded round sphere S satisfies $SL_S = L_S$. Hence S induces complex structures J on V/L_S and $\widetilde{J}$ on L_S and the conformality equation of the sphere S is

$$*\delta_S = S\delta_S = \delta_S S\,.$$

A sphere congruence assigns to each point $p \in M$ an oriented round sphere $S(p)$ in S^4. In other words, a sphere congruence is a complex structure $S \in \Gamma(\mathrm{End}(V))$ on the trivial $\mathbb{H}^2$–bundle V. Given a conformal immersion $f : M \to S^4$ with associated line bundle $L = f^*\mathcal{T}$, a sphere congruence S *envelopes* f if for all $p \in M$ the sphere $S(p)$ passes through $f(p)$ and the oriented tangent plane to f and to $S(p)$ at $f(p)$ coincide:

$$SL = L, \quad \text{and} \quad *\,\delta = S\delta = \delta S\,. \tag{3}$$

Note that S induces the complex structures $J = S_{V/L}$ and $\widetilde{J} = S|_L$ given by the conformality (2) of f.

Let $\omega \in \Omega^1(W)$ be a 1–form on M with values in a vector bundle W. If W is equipped with a complex structure $J \in \Gamma(\mathrm{End}(W))$, $J^2 = -1$, we can decompose ω into its $(1,0)$ and $(0,1)$–parts with respect to J, that is

$$\omega = \omega' + \omega''$$

where

$$\omega' = \frac{1}{2}(\omega - J*\omega), \quad \omega'' = \frac{1}{2}(\omega + J*\omega)\,.$$

We denote by $\Gamma(KW)$ and $\Gamma(\bar{K}W)$ the $(1,0)$ respectively $(0,1)$–forms with values in the complex bundle (W,J). For instance, if $f : M \to S^4$ is a conformal immersion its derivative $\delta \in \Gamma(K\,\mathrm{Hom}(L, V/L))$ is a $(1,0)$–form by (2).

A conformal immersion $f : M \to S^4$ induces [4] an elliptic first order differential operator

$$D : \Gamma(V/L) \to \Gamma(\bar{K}V/L),$$

a so–called *quaternionic holomorphic structure* on V/L given by

$$D\varphi = (\pi d\widetilde{\varphi})''\,. \tag{4}$$

Here $\widetilde{\varphi} \in \Gamma(V)$ is an arbitrary lift of $\varphi \in \Gamma(V/L)$ under π. The holomorphic structure D is well–defined since $\pi d|_L = \delta \in \Gamma(K\,\mathrm{Hom}(L, V/L))$ and thus $D\psi = (\delta\psi)'' = 0$ for $\psi \in \Gamma(L)$. We denote by

$$H^0(V/L) = \ker D$$

the space of *holomorphic sections* of V/L.

An important property of the holomorphic structure D is that f is given as a quotient of holomorphic sections and Darboux transforms of f are given by prolongations of holomorphic sections. If $W \to M$ is a quaternionic line

bundle over M we write $\widetilde{W}$ for its pull–back to the universal cover $\widetilde{M}$. A section $\varphi \in \Gamma(\widetilde{W})$ with monodromy is one which satisfies

$$\gamma^*\varphi = \varphi h_\gamma, \quad h_\gamma \in \mathbb{H}_*$$

for a representation h of the fundamental group $\pi_1(M)$ acting by deck transformations on $\widetilde{M}$.

Lemma 2.1 ([4]). *Let $f\colon M \to S^4$ be a conformal immersion, $L \subset V$ the associated line subbundle of V and $\varphi \in H^0(\widetilde{V/L})$ a non–trivial holomorphic section. Then φ has a unique lift, with respect to the projection $\pi\colon V \to V/L$, to a section $\widehat{\varphi} \in \Gamma(\widetilde{V})$ such that*

$$\pi d\widehat{\varphi} = 0\,. \tag{5}$$

Away from its (isolated) zeros the prolongation $\widehat{\varphi}$ *of φ defines a conformal map $\widehat{f}\colon \widetilde{M} \to S^4$, namely $\widehat{f}(p) = \widehat{\varphi}(p)\mathbb{H}$. If the holomorphic section $\varphi \in H^0(\widetilde{V/L})$ has monodromy then $\widehat{\varphi}$ has the same monodromy and thus $\widehat{f}$ descends to a conformal map $\widehat{f}\colon M \to S^4$.*

Definition 2.2. Let $f\colon M \to S^4$ be a conformal immersion and $L \subset V$ be the associated line subbundle of V. Conformal maps $\widehat{f}$ defined (away from isolated points) by holomorphic sections of $\widetilde{V/L}$ are called *Darboux transforms* of f.

Darboux transforms naturally generalise the classical Darboux transforms since $\widehat{f}$ is a Darboux transform of f if and only if (away from isolated points) there exists a sphere congruence S enveloping f and left–enveloping $\widehat{f}$ [4]. To say that S *left–envelopes* $\widehat{f}$ means that

$$S\widehat{L} = \widehat{L} \quad \text{and} \quad *\widehat{\delta} = S\widehat{\delta}\,, \tag{6}$$

the latter expressing only half of the enveloping condition (3).

Definition 2.3. Let $f\colon M \to S^4$ be a conformal immersion. A conformal map $f^\sharp : M \to S^4$ is called a *classical Darboux transform* of f if $f(p) \neq f^\sharp(p)$ for all $p \in M$ and if there exists a sphere congruence enveloping both f and $f^\sharp$. In this case $(f, f^\sharp)$ is called a *classical Darboux pair.*

Darboux [7] showed that $(f, f^\sharp)$ form a classical Darboux pair if and only if f and $f^\sharp$ are both *isothermic*, that is f and $f^\sharp$ allow conformal curvature line parametrisations away from umbilic points.

We now express the condition for $(f, f^\sharp)$ to be a classical Darboux pair in terms of the derivatives δ and $\delta^\sharp$ of f and $f^\sharp$. Given two surfaces $f, f^\sharp : M \to S^4$ with $f(p) \neq f^\sharp(p)$ for all $p \in M$, the trivial $\mathbb{H}^2$–bundle V splits as

$$V = L \oplus L^\sharp\,.$$

We write the trivial connection d on V in this splitting as

$$d = \begin{pmatrix} \nabla^L & \delta^\sharp \\ \delta & \nabla^\sharp \end{pmatrix},$$

where ∇^L and $\nabla^\sharp$ are connections on L and $L^\sharp$ respectively. Moreover,

$$\delta \in \Omega^1(\mathrm{Hom}(L, L^\sharp)) \quad \text{and} \quad \delta^\sharp \in \Omega^1(\mathrm{Hom}(L^\sharp, L)),$$

are the derivatives of f and $f^\sharp$ when we identify $V/L = L^\sharp$ and $V/L^\sharp = L$ via the bundle isomorphisms $\pi|_{L^\sharp} : L^\sharp \to V/L$ and $\pi^\sharp|_L : L \to V/L^\sharp$.

Lemma 2.4 ([6])**.** *Let $f, f^\sharp : M \to S^4$ be conformal immersions. Then $(f, f^\sharp)$ is a classical Darboux pair if and only if $L \oplus L^\sharp = V$, and*

$$\delta^\sharp \wedge \delta = \delta \wedge \delta^\sharp = 0. \tag{7}$$

Proof. Let $(f, f^\sharp)$ be a classical Darboux pair with enveloping sphere congruence S. Then

$$*\delta = S\delta = \delta S \quad \text{and} \quad *\,\delta^\sharp = S\delta^\sharp = \delta^\sharp S$$

say that δ and $\delta^\sharp$ have type $(1,0)$ with respect to the complex structure S and so

$$\delta \wedge \delta^\sharp = \delta^\sharp \wedge \delta = 0.$$

Conversely, the conformality equations

$$*\delta = J\delta = \delta\widetilde{J} \quad \text{and} \quad *\,\delta^\sharp = J^\sharp\delta^\sharp = \delta^\sharp\widetilde{J}^\sharp$$

for f and $f^\sharp$ together with (7) give $\widetilde{J} = J^\sharp$ and $\widetilde{J}^\sharp = J$. Hence the complex structure

$$S = \begin{pmatrix} J^\sharp & 0 \\ 0 & J \end{pmatrix}$$

expressed in the splitting $V = L \oplus L^\sharp$ envelopes f and $f^\sharp$ and therefore $(f, f^\sharp)$ form a classical Darboux pair. □

So far our considerations have been Möbius invariant. Choosing a point at infinity $\infty \in S^4$, lying neither on f nor $f^\sharp$, we can consider $f, f^\sharp \colon M \to \mathbb{H}$ as maps into Euclidean 4–space. Then we can write $f^\sharp = f + T$ with $T : M \to \mathbb{H}_*$ provided that for all $p \in M$, we have $f(p) \neq f^\sharp(p)$. We may take $\infty = e\mathbb{H}$ with $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{H}^2$ so that

$$\psi = \begin{pmatrix} f \\ 1 \end{pmatrix} \in \Gamma(L), \quad \psi^\sharp = \begin{pmatrix} f^\sharp \\ 1 \end{pmatrix} \in \Gamma(L^\sharp) \tag{8}$$

give trivialisations of L and $L^\sharp$ respectively. The derivatives of f and $f^\sharp$ in the splitting $V = L \oplus L^\sharp$ then calculate to

$$\delta\psi = \mathrm{pr}_{L^\sharp}\begin{pmatrix} df \\ 0 \end{pmatrix} = \psi^\sharp T^{-1} df \quad \text{and} \quad \delta^\sharp\psi^\sharp = \mathrm{pr}_L\begin{pmatrix} df^\sharp \\ 0 \end{pmatrix} = -\psi T^{-1} df^\sharp. \tag{9}$$

If f and $f^\sharp$ are a classical Darboux pair, then (7) yields

$$T^{-1}df^\sharp T^{-1}\wedge df = df\wedge T^{-1}df^\sharp T^{-1} = 0\,, \tag{10}$$

and thus

$$d(T^{-1}df^\sharp T^{-1}) = -T^{-1}dTT^{-1}\wedge df^\sharp T^{-1} + T^{-1}df^\sharp\wedge T^{-1}dTT^{-1} = 0\,,$$

where we used $df^\sharp = df + dT$. Therefore, locally $T^{-1}f^\sharp T^{-1} = f^d$ for a conformal immersion $f^d : M \to \mathbb{R}^4$ satifisfying

$$df\wedge df^d = df^d\wedge df = 0\,. \tag{11}$$

A conformal map $f^d\colon M\to\mathbb{R}^4$ satisfying (11) is called a *dual surface* to f. If a dual surface exists it is unique up to translation and a real scaling [11]. As we have seen any isothermic surface $f\colon M\to\mathbb{R}^4$ admits a dual surface f^d. The converse also holds as can be seen by reversing the above calculations: we obtain classical Darboux transforms $f^\sharp$ by solving the Riccati equation

$$dT = -df + Tdf^dT \tag{12}$$

and putting $f^\sharp = f + T : M \to \mathbb{R}^4$. This is a well–known description of isothermic surfaces via their dual surfaces [12].

3. Constant mean curvature surfaces

We now turn to the case when the immersion $f\colon M\to\mathbb{R}^3$ has constant mean curvature. Then f is isothermic and applying the classical Darboux transformation we obtain isothermic surfaces. These have constant mean curvature when the initial condition for the Riccati equation (12) is chosen appropriately [12]. On the other hand, a surface of constant mean curvature has an associated $\mathbb{C}_*$–family of flat connections ∇^μ. We show that the parallel sections of these connections give rise to Darboux transforms of f which we call *μ–Darboux transforms*. Only for special values of the spectral parameter μ do they become classical Darboux transforms.

We view $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ as the imaginary quaternions. Then the Gauss map $N : M \to S^2\subset\mathbb{R}^3$ of a conformal immersion $f : M\to\mathbb{R}^3$ satisfies

$$*df = Ndf = -dfN$$

and N is harmonic if and only if f has constant mean curvature [22]. The harmonicity condition for $N : M\to S^2\subset\mathbb{R}^3$ is given by

$$d(dN)'' = 0 \tag{13}$$

where $(dN)'' = \frac{1}{2}(dN + N*dN)$ is the (0,1)–part of dN with respect to N. Note that N is a complex structure since $N^2 = -1$. The splitting of dN

$$dN = (dN)' + (dN)''$$

into $(1,0)$ and $(0,1)$–parts is [6, p.40] the decomposition of the shape operator into trace and tracefree parts so that

$$(dN)' = -Hdf, \tag{14}$$

where H is the mean curvature of f. With this normalisation the mean curvature of the unit sphere with respect to the inward normal is $H = 1$. From now on we assume that we have scaled our constant mean curvature surfaces so that $H = 1$. Then the parallel constant mean curvature surface

$$g = f + N$$

satisfies $dg = (dN)''$ and thus type considerations give $dg \wedge df = df \wedge dg = 0$. In other words, if $f : M \to \mathbb{R}^3$ has constant mean curvature then the parallel surface $g = f + N$ is a dual surface of f which shows that f is isothermic. We consider $f : M \to \mathbb{R}^3$ as a conformal immersion into S^4 via

$$\begin{pmatrix} f \\ 1 \end{pmatrix} \mathbb{H} : M \to \mathbb{HP}^1 ,$$

where the point at infinity is $e\mathbb{H}$ with $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{H}^2$. The conformality of f gives complex structures J on V/L and $\widetilde{J}$ on L satisfying (2). Identifying V/L with $\underline{e\mathbb{H}}$ via the splitting $V = L \oplus \underline{e\mathbb{H}}$ these complex structures are given by

$$Je = eN \quad \text{and} \quad \widetilde{J}\psi = -\psi N \tag{15}$$

for the trivialising section $\psi = \begin{pmatrix} f \\ 1 \end{pmatrix}$ of L. In particular, for a constant mean curvature surface the complex structure $J \in \Gamma(\mathrm{End}(V/L))$ is harmonic. Let ∇ denote the trivial connection on $V/L = \underline{e\mathbb{H}}$ induced by the trivial connection d on $V = \underline{\mathbb{H}}^2$. From (15) and (13) we see that the harmonicity equation for J is

$$d^\nabla (\nabla J)'' = 0 .$$

With the notation

$$(\nabla J)' = -2 * A \quad \text{and} \quad (\nabla J)'' = 2 * Q$$

for the $(1,0)$ and $(0,1)$ parts, the harmonicity of J becomes

$$d^\nabla * A = 0 \quad \text{or equivalently} \quad d^\nabla * Q = 0 . \tag{16}$$

Since $J^2 = -1$ we see that ∇J anticommutes with J and therefore

$$*A = JA = -AJ \quad \text{and} \quad * Q = -JQ = QJ .$$

To reformulate the harmonicity of J as a $\mathbb{C}_*$–family of flat $\mathbf{SL}(2, \mathbb{C})$-connections we introduce the constant complex structure I, which is defined as right multiplication $I\varphi = \varphi i$ by the quaternion i. With this complex structure $V/L = \underline{\mathbb{C}}^2$ can be viewed as a trivial $\mathbb{C}^2$-bundle. The next lemma is a variant [9] of the well-known formulation of harmonicity in terms of families of flat connections.

Lemma 3.1. *Let $J \in \Gamma(\mathrm{End}(V/L))$ be a complex structure on V/L with flat connection ∇. Then J is harmonic if and only if the complex connections*

$$\nabla^\mu = \nabla + *A(J\frac{\mu + \mu^{-1} - 2}{2} + \frac{\mu^{-1} - \mu}{2} I)$$

on the complex bundle $(V/L, I)$ are flat for all $\mu = u + Iv \neq 0$, $u, v \in \mathbb{R}$.

Remark 3.2. *There are a number of useful ways to rewrite the family of flat connections ∇^μ. If we put*

$$a = \frac{\mu + \mu^{-1}}{2}, \quad b = \frac{\mu^{-1} - \mu}{2} I$$

then $a^2 + b^2 = 1$ and

$$\nabla^\mu = \nabla + *A(J(a-1) + b) = \nabla + (a - 1 + Jb)A\,, \tag{17}$$

where we used $*A = JA = -AJ$ and $[A, I] = 0$. On the other hand, using the type decomposition

$$A^{(1,0)} = \frac{1}{2}(A - I * A), \quad A^{(0,1)} = \frac{1}{2}(A + I * A)$$

of A with respect to the complex structure I, we obtain

$$\nabla^\mu = \nabla + (\mu - 1)A^{(1,0)} + (\mu^{-1} - 1)A^{(0,1)}\,. \tag{18}$$

Note that for $\mu \in S^1$, that is $a, b \in \mathbb{R}$, the connection ∇^μ is in fact quaternionic whereas ∇^μ is a complex connection for $\mu \notin S^1$ since the complex structure I is not quaternionic linear. Moreover, we see from (18) *that*

$$(\nabla^\mu \phi) j = \nabla^{\bar{\mu}^{-1}}(\phi j) \tag{19}$$

for $\phi \in \Gamma(V/L)$.

Proof of Lemma 3.1. We have $\nabla J = 2(*Q - *A)$, and by type considerations we see that $A \wedge Q = 0$, so

$$d^\nabla(*AJ) = (d^\nabla * A)J - *A \wedge \nabla J = (d^\nabla * A)J + 2 * A \wedge *A\,.$$

From this the curvature of ∇^μ computes to

$$R^\mu = (d^\nabla * A)(J(a-1) + b)$$

where we used $[A, I] = 0$ and $a^2 + b^2 = 1$. This shows that ∇^μ is a flat connection for every $\mu \in \mathbb{C}_*$ precisely when $d^\nabla * A = 0$, that is, if and only if J is harmonic. $\square$

The parallel sections of ∇^μ for $\mu \in \mathbb{C}_*$ can be given a geometric interpretation in terms of Darboux transforms of f. One observes from (4) that the trivial connection ∇ is compatible with the holomorphic structure D on V/L, that is $\nabla'' = D$. Since $*A = JA$, equation (17) then shows that also

$$(\nabla^\mu)'' = D\,.$$

Hence ∇^μ–parallel sections of V/L are in particular holomorphic without zeros and their prolongations give Darboux transforms $\widehat{f}$ of f defined on all of $\widetilde{M}$. Since $f(p) \neq \widehat{f}(p)$ for all $p \in \widetilde{M}$ we have the splitting $L \oplus \widehat{L} = V$.

Definition 3.3. Let $f\colon M \to \mathbb{R}^3$ be a constant mean curvature surface. The Darboux transforms $\widehat{f}\colon \widetilde{M} \to S^4$ given by sections of $\widetilde{V/L}$ that are parallel with respect to ∇^μ for $\mu \in \mathbb{C}_*$ are called *μ–Darboux transforms* of f.

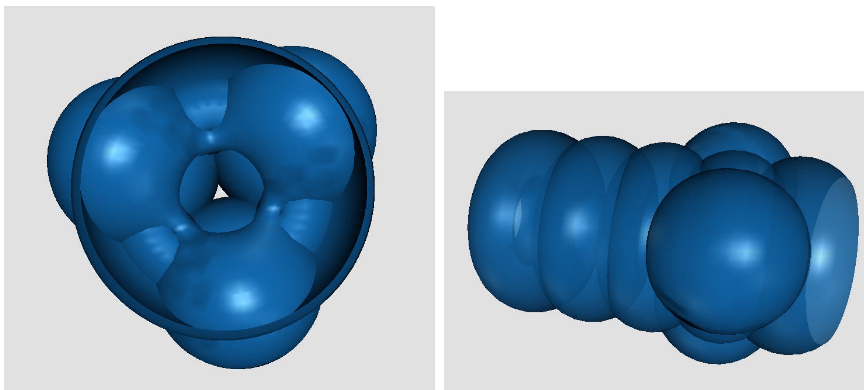

FIGURE 1. Closed μ–Darboux transform of a nodoid.

If M has topology it is generally difficult to decide whether a constant mean curvature surface f has a μ–Darboux transform $\widehat{f}$ defined on M rather than on its universal covering $\widetilde{M}$. If M is a 2–torus this question leads to the notion of the spectral curve which we will address in section 4. For now we are only interested in the local properties of μ–Darboux transforms $\widehat{f}$ and assume that M is simply connected. Then there is a $\mathbb{CP}^1$–worth of ∇^μ–parallel sections for each $\mu \in \mathbb{C}_*$ and thus the space of μ–Darboux transforms is parameterised by $\mathbb{C}_* \times \mathbb{CP}^1$.

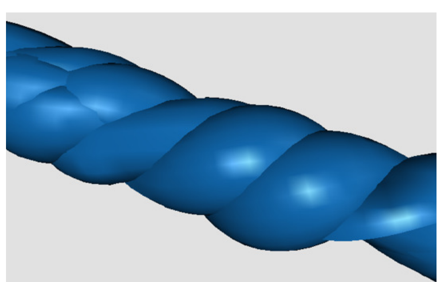

FIGURE 2. Non-closed μ–Darboux transform of an unduloid.

To better understand the geometry of μ-Darboux transforms we reinterpret the prolongation in terms of a bundle homomorphism, the *prolongation map*.

Let $\varphi \in \Gamma(V/L)$ be a ∇^μ–parallel section. The splitting $V = L \oplus \underline{e\mathbb{H}}$ identifies $V/L = \underline{e\mathbb{H}}$ so that the prolongation

$$\widehat{\varphi} = \varphi + \widehat{B}\varphi \tag{20}$$

of the nowhere vanishing section φ defines an $\mathbb{H}$–linear bundle map

$$\widehat{B} : \underline{e\mathbb{H}} \to L\,.$$

From $0 = \pi\nabla\widehat{\varphi}$ and $\nabla^\mu\varphi = 0$ we deduce

$$\delta\widehat{B}\varphi = *A(J(a-1)+b)\varphi\,. \tag{21}$$

Here $a = \frac{\mu+\mu^{-1}}{2}$, $b = \frac{\mu^{-1}-\mu}{2}I$ for $\mu \in \mathbb{C}_*$ with respect to the constant complex structure I on V/L given by right multiplication by the quaternion i. We denote by $\widehat{I}$ the quaternionic linear complex structure on V/L given by the quaternionic linear extension of I on φ, that is, $\widehat{I}\varphi = \varphi i$. Furthermore, $\widehat{a}, \widehat{b} \in \Gamma(\mathrm{End}(V/L)$ denote the quaternionic linear bundle maps obtained from a, b by replacing I with $\widehat{I}$. Then

$$\delta\widehat{B} = *A(J(\widehat{a}-1)+\widehat{b}) \in \Gamma(\mathrm{End}(V/L)) \tag{22}$$

and

$$\widehat{\nabla}^\mu = \nabla + *A(J(\widehat{a}-1)+\widehat{b}) \tag{23}$$

is a family of flat quaternionic connections on V/L with

$$\widehat{\nabla}^\mu\widehat{I} = 0\,. \tag{24}$$

Note that $\widehat{I}$ and therefore $\widehat{\nabla}^\mu$ depend on the choice of the ∇^μ–parallel section φ. We indicate this dependence by decorating φ dependent quantities by the "hat" symbol. In what follows we abbreviate

$$\widehat{C} = J(\widehat{a}-1)+\widehat{b} \tag{25}$$

and thus (22) becomes

$$\delta\widehat{B} = *A\widehat{C}\,. \tag{26}$$

In order to see that a μ–Darboux transform $\widehat{f}$ of a constant mean curvature surface f is up to translation in $\mathbb{R}^4$ a constant mean curvature surface in $\mathbb{R}^3$, we need to compare geometric data of f and $\widehat{f}$ with respect to the same choice of point at infinity $\infty = e\mathbb{H}$. For that we need to see that $\widehat{f}$ does not pass through ∞. Since f is an immersion its derivative δ has no zeros, so as δ and $*A$ have the same type,

$$*\,A = \delta R \tag{27}$$

for $R \in \Gamma(\mathrm{Hom}(V/L, L))$. Evaluating both sides of (27) on the constant section e we obtain $Re = \frac{1}{2}\psi$ where we recall (14), (15) and the trivialising section $\psi \in \Gamma(L)$ from (8). This shows that R is nowhere vanishing and parallel with respect to the connection induced by ∇ on V/L and ∇^L on L. Now $\widehat{f}$ passes through ∞ if and only if the prolongation $\widehat{\varphi} = \varphi + \widehat{B}\varphi$

of the ∇^μ parallel section $\varphi \in \Gamma(V/L)$ giving rise to $\widehat{f}$ has $\widehat{\varphi}(p) \in e\mathbb{H}$ and thus $\widehat{B}(p) = 0$ for some $p \in M$. But δ and thus by (27) also $*A$ have no zeros by the assumption that f is immersed. Therefore (26) shows that $\widehat{C} = J(\widehat{a}-1)+\widehat{b}$ has to vanish at p which implies $\mu = 1$. In this case $\widehat{\nabla}^\mu = \nabla$, the prolongation $\widehat{\varphi}$ is constant and $\widehat{f}$ is the point ∞. In the considerations to follow, we always assume $\mu \neq 1$, and in particular, both $\widehat{B}$ and $\widehat{C}$ are nowhere vanishing.

Lemma 3.4. *Let $f : M \to \mathbb{R}^3$ be a conformal immersion of constant mean curvature with $*\delta = J\delta = \delta\widetilde{J}$ and $\widehat{f} : M \to S^4$ a non–constant μ–Darboux transform of f. Then the derivative of $\widehat{f}$, expressed in the splitting in $V = \widehat{L} \oplus \underline{e\mathbb{H}}$, is given by*

$$\widehat{\delta}\widehat{R} = *\widehat{A}\,. \tag{28}$$

Here $\widehat{R} = \widehat{C}^{-1} + R$ and $-2\widehat{A} = \frac{1}{2}(\nabla\widehat{J} - \widehat{J} * \nabla\widehat{J})$ is the $(1,0)$–part of the derivative of the complex structure

$$\widehat{J} = -\widehat{C}^{-1}J\widehat{C}\,.$$

Moreover, $\widehat{J}$ is harmonic and $\widehat{f}$ is conformal with $*\widehat{\delta} = \widehat{J}\widehat{\delta} = \widehat{\delta}\widetilde{\widehat{J}}$ where $\widetilde{\widehat{J}} = -\widehat{R}\widehat{J}\widehat{R}^{-1}$.

Proof. Let φ be a ∇^μ parallel section, $\widehat{\varphi} = \varphi + \widehat{B}\varphi$ its prolongation, and $\widehat{L} = \widehat{\varphi}\mathbb{H}$ the corresponding μ–Darboux transform. Using (26) and the decomposition

$$d = \begin{pmatrix} \nabla^L & 0 \\ \delta & \nabla \end{pmatrix}$$

of d in the splitting $V = L \oplus \underline{e\mathbb{H}}$ we get

$$d\widehat{\varphi} = \nabla\varphi + \nabla^L(\widehat{B}\varphi) + \delta(\widehat{B}\varphi) = (\nabla\widehat{B})\varphi - \widehat{B}\delta\widehat{B}\varphi\,.$$

For $\phi \in L$ we have

$$\phi = (\phi + \widehat{B}^{-1}\phi) - \widehat{B}^{-1}\phi \in \widehat{L} \oplus \underline{e\mathbb{H}}\,,$$

which shows that $-\widehat{B}^{-1} \in \operatorname{Hom}(L, \underline{e\mathbb{H}})$ is the projection of L onto $\underline{e\mathbb{H}}$. Calculating $\widehat{\delta}\widehat{\varphi} = \pi_{\widehat{L}} d\widehat{\varphi}$ we obtain with $d\widehat{\varphi} \in \Omega^1(L)$ that

$$\widehat{\delta} = (-\widehat{B}^{-1}(\nabla\widehat{B}) + \delta\widehat{B})(1 + \widehat{B})^{-1}\,, \tag{29}$$

where we also used $\widehat{\varphi} = (1 + \widehat{B})\varphi$. Recalling (26) and (27) and the fact that R is parallel, we get

$$-\widehat{B}^{-1}(\nabla\widehat{B}) + \delta\widehat{B} = -\widehat{C}^{-1}\nabla\widehat{C} + *A\widehat{C}\,. \tag{30}$$

Furthermore, since $\widehat{\nabla}^\mu\widehat{I} = 0$ we have $\nabla\widehat{I} = -[*A\widehat{C}, \widehat{I}]$ and

$$\nabla\widehat{C} \;=\; 2 * Q(\widehat{a} - 1) + \widehat{C} * A\widehat{C} + *A(-2(\widehat{a} - 1) + J\widehat{C}(\widehat{a} - 1) - \widehat{C}\widehat{b})\,.$$

But since $\widehat{a}^2+\widehat{b}^2 = 1$ and $\widehat{C} = J(\widehat{a}-1)+\widehat{b}$ we have $-2(\widehat{a}-1)+J\widehat{C}(\widehat{a}-1)-\widehat{C}\widehat{b} = 0$, or equivalently

$$\widehat{J} = \frac{\widehat{b}}{1-\widehat{a}} - 2\widehat{C}^{-1}\,. \tag{31}$$

Thus we conclude that

$$\nabla\widehat{C} = 2 * Q(\widehat{a}-1) + \widehat{C} * A\widehat{C}\,. \tag{32}$$

This Riccati type equation together with (29) and (30) yields

$$\widehat{\delta} = -2\widehat{C}^{-1} * Q(\widehat{a}-1)(1+\widehat{B})^{-1}\,, \tag{33}$$

and, since $*Q = -J * Q$, we also have $*\widehat{\delta} = \widetilde{J}\widehat{\delta}$ with $\widetilde{J} = -\widehat{C}^{-1}J\widehat{C}$.

On the other hand, the derivative of $\widehat{J}$ computes to

$$\nabla\widehat{J} = \widehat{C}^{-1}\left(2 * Q\big(J(\widehat{a}-1) - (\widehat{a}-1)\widehat{J}\big) - \big(\widehat{C}J + J\widehat{C}\big) * A\widehat{C} - (\nabla J)\widehat{C}\right)\,,$$

so that its $(1,0)$–part with respect to $\widehat{J}$ is given by

$$(\nabla\widehat{J})' = -2\widehat{C}^{-1} * Q\left((\widehat{a}-1)\widehat{J} + \widehat{b}\right)\,.$$

Using (31) we thus obtain

$$* \widehat{A} = -\frac{1}{2}(\nabla\widehat{J})' = -2\widehat{C}^{-1} * Q(\widehat{a}-1)\widehat{C}^{-1}\,. \tag{34}$$

Comparison with (33) gives together with (26) and (27)

$$\widehat{\delta}\widehat{R} = *\widehat{A}$$

where $\widehat{R} = \widehat{C}^{-1} + R$, and from $*\widehat{A} = \widehat{J}\widehat{A} = -\widehat{A}\widehat{J}$ we also see $*\widehat{\delta} = \widetilde{J}\widehat{\delta} = \widehat{\delta}\widetilde{\widehat{J}}$. Finally, by (34) and the Riccati type equation (32) we see that

$$*\widehat{A} = *A + \nabla\widehat{C}^{-1}$$

is d^∇-closed since $d^\nabla * A = 0$ by the harmonicity of J.

□

Notice that equation (33) shows that a μ–Darboux transform is constant if and only if $Q = 0$ or $\widehat{a} = 1$. The first condition means that the Gauss map N is holomorphic and thus $f(M)$ is contained in a round sphere whilst the second is equivalent to $\mu = 1$.

Corollary 3.5. *If $f(M)$ is not contained in a round sphere, then a μ-Darboux transform $\widehat{f}$ of f is constant if and only if $\mu = 1$. In this case, $\widehat{f} = \infty$.*

Using Lemma 3.4 we arrive at our first main result that every μ-Darboux transform of a constant mean curvature surface has constant mean curvature.

Theorem 3.6. *Let $f\colon M \to \mathbb{R}^3$ be a surface of constant mean curvature $H = 1$. Then every μ-Darboux transform $\widehat{f}$ of f has constant real part and when $\widehat{f}$ is not a point, $\operatorname{Im}(\widehat{f})$ is a constant mean curvature surface in $\mathbb{R}^3$ with $\widehat{H} = 1$. In particular, for each $\mu = e^{i\theta} \in S^1 \setminus \{1\}$ there is a unique μ-Darboux transform of f, given by $\widehat{f} = g + \cot\frac{\theta}{2}$, where $g = f + N$ is the parallel constant mean curvature surface of f.*

Proof. Let φ be a ∇^μ–parallel section and $\widehat{f}$ the associated μ–Darboux transform for $\mu \neq 1$. Writing $\widehat{C}^{-1}e = \frac{1}{2}e\widehat{T}$ with $\widehat{T} : M \to \mathbb{H}_*$ we have

$$\widehat{R}e = (R + \widehat{C}^{-1})e \;= \frac{1}{2}\begin{pmatrix} f + \widehat{T} \\ 1 \end{pmatrix} \in \Gamma(\widehat{L}) \tag{35}$$

so that $\widehat{f} = f + \widehat{T}$. Moreover, the complex structures $\widehat{J} = -\widehat{C}^{-1}J\widehat{C}$ and $\widetilde{\widehat{J}} = -\widehat{R}\widehat{J}\widehat{R}^{-1}$ satisfy

$$\widehat{J}e = e\widehat{N} \quad \text{and} \quad \widetilde{\widehat{J}}\widehat{\psi} = -\widehat{\psi}\widehat{N} \tag{36}$$

with $\widehat{N} = -\widehat{T}N\widehat{T}^{-1}$ and $\widehat{\psi} = 2\widehat{R}e$. In particular, $*\widehat{\delta}\widehat{\psi} = \widehat{J}\widehat{\delta}\widehat{\psi} = \widehat{\delta}\widetilde{\widehat{J}}\widehat{\psi}$ reads in coordinates as

$$*d\widehat{f} = \widehat{N}d\widehat{f} = -d\widehat{f}\widehat{N}$$

so that $d\widehat{f}$ takes values in $\mathbb{R}^3$, and $\widehat{f}$ has constant real part. Therefore, $\widehat{N}$ is the Gauss map of $\operatorname{Im}(\widehat{f})$. From Lemma 3.4 we know that $\widehat{J}$ is harmonic and thus $\operatorname{Im}(\widehat{f})$ has constant mean curvature. In fact, (28) shows that the derivative of $\widehat{f}$, and thus of $\operatorname{Im}(\widehat{f})$, is $e\,d\widehat{f} = -e(d\widehat{N})'$ so that the mean curvature of $\operatorname{Im}(\widehat{f})$ is $\widehat{H} = 1$. Writing $\varphi = e\alpha$, then evaluating $\widehat{C}$ on e gives

$$e\widehat{T}^{-1} = \frac{1}{2}\widehat{C}e = \frac{1}{2}e(N\alpha(a-1)\alpha^{-1} + \alpha b\alpha^{-1}) \tag{37}$$

where μ, $a = \frac{\mu+\mu^{-1}}{2}$ and $b = \frac{\mu-\mu^{-1}}{2i}$ are viewed as complex numbers. This shows for $\mu \in S^1$, that is $a, b \in \mathbb{R}$, that the μ-Darboux transform is given by

$$\widehat{f} = f + N + \frac{b}{1-a} = g + \cot\tfrac{\theta}{2}$$

and hence is a translate of the parallel constant mean curvature surface $g = f + N$ and is independent of the parallel section φ. □

Remark 3.7. *Observe that* (37) *also shows that a μ–Darboux transform $\widehat{f}$ of f has vanishing real part for $\mu \in \mathbb{R}_*$. Furthermore, for $\mu \in \mathbb{C}_* \setminus (S^1 \cup \mathbb{R}_*)$ the real part of a μ–Darboux transform is* $\operatorname{Re}(\widehat{f}) = \frac{\operatorname{Im} a}{\operatorname{Im}\left((a-1)\bar{b}^{-1}\right)}$ *where $a = \frac{\mu+\mu^{-1}}{2}$ and $b = \frac{\mu^{-1}-\mu}{2}I$.*

Remark 3.8. *There are versions of Lemma 3.4 for other classes of immersions given by a harmonicity condition. Analogues of Theorem 3.6 hold for Hamiltonian stationary Lagrangian surfaces* [18]*, and for (constrained) Willmore surfaces in the 4–sphere* [3].

We conclude this section by determining which μ–Darboux transforms $\widehat{f}$ of a constant mean curvature immersion f are classical. Since a μ–Darboux transform $\widehat{f}$ is a Darboux transform there is a sphere congruence S enveloping f and left–enveloping $\widehat{f}$, hence satisfying (3) and (6). From arguments analogous to those in the proof of Lemma 2.4 (see also [4]) it follows that these enveloping conditions are equivalently described by $\delta \wedge \widehat{\delta} = 0$. Therefore, in order to see which μ–Darboux transforms are classical, we need by Lemma 2.4 to investigate the condition $\widehat{\delta} \wedge \delta = 0$ in the splitting $V = L \oplus \widehat{L}$.

Theorem 3.9. *A non–constant μ–Darboux transform $\widehat{f}\colon M \to \mathbb{R}^4$ of a constant mean curvature surface $f : M \to \mathbb{R}^3$ is a classical Darboux transform of f if and only if $\mu \in \mathbb{R}_* \cup S^1 \setminus \{1\}$.*

Note that by Remark 3.7 the μ-Darboux transform $\widehat{f}$ takes values in a parallel translated $\mathbb{R}^3 \subset \mathbb{R}^4$ of distance $\cot \frac{\theta}{2}$ where $\mu = e^{i\theta}$. In particular, $\widehat{f}$ takes values in the same $\mathbb{R}^3$ as f if and only if $\mu \in \mathbb{R} \setminus \{0, 1\}$.

Proof. Let $\varphi \in \Gamma(V/L)$ be a parallel section of ∇^μ and $\widehat{f}$ the Darboux transform given by φ. From the definition of the prolongation map $\widehat{B}$ we see that the decomposition of $\phi \in \underline{e\mathbb{H}}$ with respect to the splitting $V = L \oplus \widehat{L}$ is

$$\phi = (-\widehat{B}\phi) + (1 + \widehat{B})\phi \in L \oplus \widehat{L}\,.$$

Therefore, using equations (27) and (33) the derivatives of f and $\widehat{f}$ with respect to this splitting are given by

$$\delta R = (1 + \widehat{B}) * A \quad \text{ and } \quad \widehat{\delta}(1 + \widehat{B}) = 2R * Q(\widehat{a} - 1)\,.$$

Since $\widehat{f}$ is not constant $\widehat{\delta}$ has only isolated zeros and so does Q. Then

$$2R * Q(\widehat{a} - 1) \wedge *A = \widehat{\delta} \wedge \delta R = 0$$

if and only if $*Q(\widehat{a} - 1) = Q(\widehat{a} - 1)J$ where we used $*A = JA$. Since Q has isolated zeros this last is equivalent to $[J, \widehat{a} - 1] = 0$, that is

$$(\operatorname{Im} \widehat{a})[J, \widehat{I}] = 0\,. \tag{38}$$

However the two complex structures J and $\widehat{I}$ commute only if $\widehat{I} = \pm J$. By (24) and (23) this implies

$$0 = \widehat{\nabla}^\mu J = 2(*Q - *A) + [*A\widehat{C}, J]\,.$$

Considering the $(0, 1)$–part, this yields

$$*Q = 0\,,$$

contradicting the assumption that $\widehat{f}$ is not constant. Therefore, (38) is equivalent to $\operatorname{Im} \widehat{a} = 0$, which shows that a non–constant μ–Darboux transform is a classical Darboux transform if and only if $\mu \in \mathbb{R}_* \cup S^1 \setminus \{1\}$. $\square$

Since the real part of a μ–Darboux transform $\widehat{f}$ of a constant mean curvature surface $f\colon M \to \mathbb{R}^3$ is constant only its imaginary part $\mathrm{Im}\,(\widehat{f})$ is geometrically relevant.

Theorem 3.10. *Let $f\colon M \to \mathbb{R}^3$ be a constant mean curvature surface and $\widehat{f}$ a μ-Darboux transform. Then $\widehat{f}$ is a classical Darboux transform if and only if $\mathrm{Im}\,(\widehat{f})$ is.*

Proof. If $\widehat{f}$ is a classical Darboux transform then by Theorem 3.6 and Remark 3.7 either $\widehat{f}$ has vanishing real part or it is the parallel constant mean curvature surface of f up to a real translation. Therefore $\mathrm{Im}\,\widehat{f}$ is also a classical Darboux transform. On the other hand, let us assume that $\mathrm{Im}\,(\widehat{f})$ is a classical Darboux transform. Then it follows from (10), (36) and the fact that $\widehat{f}$ and $\mathrm{Im}\,(\widehat{f})$ differ by a real constant that

$$\widehat{N} = -\widehat{T}N\widehat{T}^{-1} = -\mathrm{Im}\,(\widehat{T})N\,\mathrm{Im}\,(\widehat{T})^{-1}\,,$$

where $\widehat{T} = \widehat{f} - f$. Assuming $\mathrm{Re}\,(\widehat{T}) = \mathrm{Re}\,(\widehat{f}) \neq 0$ we conclude $\mathrm{Im}\,(\widehat{T}) = rN$ with $r\colon M \to \mathbb{R}$ and thus $\widehat{N} = -N$. From Theorem 3.6 we know that $\mathrm{Im}\,(\widehat{f})$ is a constant mean curvature surface whose Gauss map is $-N$ and therefore $\mathrm{Im}\,(\widehat{f})$ is the parallel constant mean curvature surface of f. Now (31) and (37) imply that $\mathrm{Re}\,(\widehat{T}) = \frac{\widehat{b}}{1-\widehat{a}}$ which, using $\widehat{a}^2 + \widehat{b}^2 = 1$, shows that $\widehat{a} = \frac{\mathrm{Re}\,(\widehat{T})^2-1}{\mathrm{Re}\,(\widehat{T})^2+1} \in \mathbb{R}$ and therefore $\mu \in S^1$. But then Theorem 3.9 implies that $\widehat{f}$ is a classical Darboux transform. $\square$

So far we have seen that μ–Darboux transforms are classical Darboux transforms if and only if $\mu \in \mathbb{R}_* \cup S^1$. We now turn to the question of which classical Darboux transforms are μ–Darboux transforms.

Theorem 3.11. *A classical Darboux transform $f^\sharp\colon M \to \mathbb{R}^4$ of a constant mean curvature surface $f\colon M \to \mathbb{R}^3$ is a μ–Darboux transform if and only if $T = f^\sharp - f$ satisfies the Riccati equation*

$$dT = rTdgT - df\,, \quad (T-N)^2 = r^{-1} - 1 \quad r \in \mathbb{R}\setminus\{0,1\}\,. \tag{39}$$

In this case Theorem 3.9 shows that $\mu \in \mathbb{R}_* \cup S^1 \setminus \{1\}$. When $\mu \in S^1$ the classical Darboux transform $f^\sharp$ is a translate of the parallel constant mean curvature surface $g = f + N$ and for $\mu \in \mathbb{R}_*$ it is a constant mean curvature surface in $\mathbb{R}^3$.

Proof. Let $f^\sharp : M \to \mathbb{R}^4$ be a μ–Darboux transform for $\mu \in \mathbb{R}_* \cup S^1 \setminus \{1\}$. With (32) we see that $T = f^\sharp - f$ satisfies the Riccati equation with $r = \frac{1-\widehat{a}}{2} \in \mathbb{R}$ since $\mu \in \mathbb{R}_* \cup S^1$. Now (37) gives

$$((Tr)^{-1} + N)^2 = \frac{\widehat{b}^2}{(1-\widehat{a})^2} = r^{-1} - 1\,,$$

where we used $\widehat{a}^2 + \widehat{b}^2 = 1$. Since $r \in \mathbb{R}_*$ this equation is equivalent to $(T-N)^2 = r^{-1} - 1$.

Conversely, let T be a solution of the Riccati equation with $r \in \mathbb{R}_*, r \neq 1$ and (39). We put

$$\widehat{a} = 1 - 2r \quad \text{and} \quad \widehat{b} = 2(T^{-1} + Nr)\,.$$

If $r \in (0,1)$ then $\widehat{a}^2 + \widehat{b}^2 = 1$ and $\widehat{a} \in \mathbb{R}$ imply $\widehat{b} \in \mathbb{R}$. In particular, $\mu = \widehat{a} + I\widehat{b} \in S^1$, and $T = N + \frac{\widehat{b}}{1-\widehat{a}}$. Thus, $\widehat{f} = f + T$ is a μ–Darboux transform of f with $\mu \in S^1$.

If $r \in \mathbb{R} \setminus [0,1]$ then $\widehat{a}^2 + \widehat{b}^2 = 1$, $|\widehat{a}| > 1$, and $\widehat{a} \in \mathbb{R}$ imply that $\widehat{b}$ has values in the imaginary quaternions. Moreover, the quaternionic connection $\widehat{\nabla} = \nabla + \omega$, where $\omega \in \Omega^1(\underline{e\mathbb{H}})$ is defined by

$$\omega e = edfT^{-1}\,,$$

is flat since T satisfies the Riccati equation. Let $\varphi = e\alpha$ be a $\widehat{\nabla}$–parallel section, that is $(d\alpha)\alpha^{-1} = -dfT^{-1}$. Using again the Riccati equation together with (39) we see

$$d(\alpha^{-1}\widehat{b}\alpha) = 2\alpha^{-1}\left([dfT^{-1}, T^{-1} + Nr] + d(T^{-1}) + dNr\right)\alpha = 0\,.$$

Thus we may assume after scaling φ by a quaternion that $\widehat{b}\varphi = \varphi b_0 i$ with $b_0 \in \mathbb{R}$ since $\widehat{b}^2 < 0$. In particular,

$$(J(\widehat{a} - 1) + Ib_0)\varphi = 2\varphi T^{-1}$$

so that φ is a parallel section of the complex connection ∇^μ for $\mu = \widehat{a} - b_0 \in \mathbb{R}$. The prolongation of φ gives a μ–Darboux transform of f which is exactly $f^\sharp = f + T$. □

In [12] it is shown that a classical Darboux transform $f^\sharp\colon M \to \mathbb{R}^3$ of a constant mean curvature surface $f\colon M \to \mathbb{R}^3$ has constant mean curvature if and only if $T = f^\sharp - f$ satisfies (39) with $r \in \mathbb{R} \setminus \{0,1\}$. Therefore we obtain:

Corollary 3.12. *Let $f : M \to \mathbb{R}^3$ be a constant mean curvature surface. The classical Darboux transforms $f^\sharp : M \to \mathbb{R}^3$ of f with constant mean curvature are exactly the μ–Darboux transforms of f with $\mu \in \mathbb{R} \setminus \{0,1\}$.*

4. The eigenline Spectral Curve

Thus far we have discussed the local properties of μ–Darboux transforms. We now turn our attention to the question of global existence of solutions: given a constant mean curvature surface $f : M \to \mathbb{R}^3$ where M has topology, is there a μ–Darboux transform $\widehat{f} : M \to S^4$ of f defined on M rather than on the universal cover $\widetilde{M}$ of M? In general, this question is hard to decide, however in the case of a constant mean curvature torus $f : T^2 \to \mathbb{R}^3$ we shall see that there are many such transformations. In fact we will show

that the space of μ–Darboux transforms of f is given by Hitchin's spectral curve [13] together with finitely many complex projective lines. Henceforth when we refer to Darboux transforms of a constant mean curvature torus we will always assume that they are defined on T^2, rather than merely on its universal cover.

As many authors have noted (see eg. [21, 26, 13]), a harmonic map $N\colon M \to S^2$ from a Riemann surface M into the 2-sphere gives rise to a family of flat connections ∇^ζ on a trivial $\mathbb{C}^2$-bundle. In the quaternionic setting we view our trivial complex bundle as a quaternionic line bundle W over M with a trivial connection ∇. Choosing a ∇–parallel section ϕ, the harmonic map N gives rise to a harmonic complex structure J on W by setting $J\phi = \phi N$. Then the endomorphism J is parallel with respect to the pull back

$$\widetilde{\nabla} = \nabla + \frac{1}{2}J^{-1}\nabla J$$

under N of the Levi-Civita connection of S^2. Writing $\omega = -\frac{1}{2}J^{-1}\nabla J$, the family of flat connections is given by

$$\nabla^\zeta = \widetilde{\nabla} + \zeta\omega^{(1,0)} + \zeta^{-1}\omega^{(0,1)}, \quad \zeta \in \mathbb{C}_*\,, \tag{40}$$

where the type decomposition of ω is with respect to the constant complex structure I on W which is right multiplication by the quaternion i. However, in Lemma 3.1 we encoded the harmonicity of a complex structure J on a trivial quaternionic line bundle as the requirement that the connections (17)

$$\nabla^\mu = \widetilde{\nabla} + (a + Jb)A + Q, \quad \mu = a + bI \in \mathbb{C}_*$$

have zero curvature. Recall here that $A = \frac{1}{4}(J\nabla J + *\nabla J)$ and $Q = \frac{1}{4}(J\nabla J - *\nabla J)$ give the type decomposition of ω with respect to the complex structure J on W.

Lemma 4.1. *The families of flat connections ∇^μ and ∇^ζ defined above are gauge equivalent, with $\mu = \zeta^2$.*

Proof. Write $\zeta = u + vI$ with $u = \frac{\zeta+\zeta^{-1}}{2}$, $v = \frac{\zeta^{-1}-\zeta}{2}I$ so that $u^2 + v^2 = 1$. Then (40) shows that

$$\nabla^\zeta \;=\; \widetilde{\nabla} + (u + vJ)A + (u - vJ)Q.$$

Define $\mu = \zeta^2$, and put $\lambda = a + bJ$. Furthermore, denote by $\lambda^{\frac{1}{2}} = u + vJ$ the choice of square root whose coefficients agree with those of ζ. Since J is parallel with respect to $\widetilde{\nabla}$ we see

$$\lambda^{\frac{1}{4}} \cdot \nabla^\zeta = \lambda^{\frac{1}{4}}(\widetilde{\nabla} + \lambda^{\frac{1}{2}}A + \lambda^{-\frac{1}{2}}Q)\lambda^{-\frac{1}{4}} = \widetilde{\nabla} + \lambda A + Q = \nabla^\mu.$$

Note that both choices of square root of $\lambda^{\frac{1}{2}}$ produce the same gauge. □

We now return to those harmonic maps N arising as Gauss maps of constant mean curvature immersions $f\colon M \to \operatorname{Im}\mathbb{H} \simeq \mathbb{R}^3$. In this case $W = V/L$ where $L \subset V$ is the line subbundle of the trivial $\mathbb{H}^2$-bundle V given by

the immersion f. As explained earlier the point at infinity $e\mathbb{H}$ defines a splitting $V = L \oplus \underline{e\mathbb{H}}$ which induces on $V/L = \underline{e\mathbb{H}}$ a trivial connection ∇ with parallel section $\phi = e$. Specialising to the case when $M = T^2$ is a torus for $\gamma \in \pi_1(T^2, p)$ let

$$H^\mu_\gamma(p) \in \mathbf{SL}(2, \mathbb{C})$$

be the holonomy of ∇^μ about γ with base point $p \in T^2$. For generic $\mu \in \mathbb{C}_*$ there is a unique pair of lines $\mathcal{E}_\mu(p)$ and $\widetilde{\mathcal{E}}_\mu(p)$ in $(V/L)_p$ which vary holomorphically in μ and are eigenlines for $H^\mu_\gamma(p)$. Since the fundamental group of the torus is abelian, the holonomy matrices for different generators $\gamma \in \pi_1(T^2, p)$ commute, and so the eigenlines do not depend upon the choice of γ.

We will define the eigenline spectral curve of f by taking the minimally branched 2-sheeted cover of $\mathbb{CP}^1$ on which these eigenlines, and their limits as $\mu \to 0, \infty$, are well-defined. To determine the branching over $\mu = 0, \infty$, we investigate the limiting behaviour of the eigenlines and eigenvalues of the holonomy. Denote by $E(p)$ the i-eigenspace of $J(p)$, then $E(p)j$ is the $(-i)$-eigenspace.

Theorem 4.2. (i) *The (common) eigenlines of the holonomy $H^\mu(p)$ have the following holomorphic limits:*

$$\lim_{\mu\to 0} \mathcal{E}_\mu(p) = \lim_{\mu\to 0} \widetilde{\mathcal{E}}_\mu(p) = E(p)j$$
$$\lim_{\mu\to \infty} \mathcal{E}_\mu(p) = \lim_{\mu\to \infty} \widetilde{\mathcal{E}}_\mu(p) = E(p).$$

(ii) *These eigenlines each agree in the limit only to first order.*

(iii) *For $\gamma \in \pi_1(T^2)$, denote by $h_\gamma(\zeta)$ the eigenvalues of the holonomy H^μ_γ, where $\zeta^2 = \mu$. There is a punctured neighbourhood U of $\infty \in \mathbb{CP}^1$ and $w_{-1} \in \mathbb{C}_*$, $u_i : \pi_1(T^2) \to \mathbb{C}$ satisfying*

$$\log h_\gamma(\zeta) = \pm(w_{-1}\gamma\zeta + u_0(\gamma) + u_1(\gamma)\zeta^{-1} + \cdots), \quad \textit{for } \zeta \in U$$
$$\log h_\gamma(\zeta) = \mp(\bar{w}_{-1}\gamma\zeta^{-1} + \bar{u}_0(\gamma) + \bar{u}_1(\gamma)\zeta + \cdots), \quad \textit{for } \bar{\zeta}^{-1} \in U$$

where we interpret γ as a complex number.

Proof. We first give the proof of (i), which will include the statement and proof of Lemma 4.3 below.

From [9, Sec. 6.3] there are only finitely many μ for which the holonomy of ∇^μ has just one eigenspace. Thus we may let U be a punctured neighbourhood of $\infty \in \mathbb{CP}^1$ consisting of μ for which the eigenlines of the holonomy H^μ are distinct. The $(0,1)$ part of ∇^μ with respect to the complex structure I gives by (18) the complex holomorphic structure

$$(\nabla^\mu)^{(0,1)} = \nabla^{(0,1)} + (\mu^{-1} - 1)A^{(0,1)}$$

on V/L resulting in a complex rank two holomorphic vector bundle over $T^2 \times (U \cup \{\infty\})$. The restriction $V/L_{(\cdot,\mu)}$ is a bundle over T^2 with holomorphic structure $(\nabla^\mu)^{(0,1)}$.

The argument of [13, Prop. 3.5] may be applied here to show that U can be chosen so that for each $\mu \in U$,

$$\dim H^0\left(\mathrm{End}_0(V/L_{(\cdot,\mu)})\right) = 1\,,$$

and all holomorphic sections are in fact parallel with respect to ∇^μ. Here End_0 denotes the bundle of trace free endomorphisms. Thus we can take a family of non-trivial holomorphic sections

$$\Psi(p,\mu) = \Psi_0(p) + \Psi_1(p)\mu^{-1} + \cdots$$

of $\mathrm{End}_0(V/L_{(\cdot,\mu)})$ for $\mu \in U \cup \{\infty\}$. Since Ψ is also ∇^μ–parallel, we may use it to investigate the limiting behaviour of the holonomy. We have

$$\left(\nabla^{(1,0)} + (\mu - 1)\,\mathrm{ad}_{A^{(1,0)}}\right)(\Psi_0 + \Psi_1\mu^{-1} + \cdots) = 0\,. \tag{41}$$

In particular

$$[A^{(1,0)}, \Psi_0] = 0 \tag{42}$$

and

$$\nabla^{(1,0)}\Psi_0 + [A^{(1,0)}, \Psi_1] = 0\,. \tag{43}$$

In fact $A^{(1,0)}$ and Ψ_0 are related by a *global* holomorphic differential, as we now show.

Lemma 4.3. *Let dz be a global nontrivial holomorphic differential on the torus T^2. Then $\Psi(p,\mu) = \Psi_0(p) + \Psi_1(p)\mu^{-1} + \cdots$ may be chosen so that*

$$A^{(1,0)} = \Psi_0 dz.$$

Proof. We first show that $A^{(1,0)}$ is nowhere vanishing. Since $-2A^{(1,0)} = (I + J) * A$ it suffices to show that $\mathrm{im}(*A) \nsubseteq Ej$. But since $*A$ is right $\mathbb{H}$-linear and from (27) is nowhere vanishing, $\mathrm{im}(*A) \subseteq Ej$ is impossible. Using (42), for any choice of parallel endomorphism Ψ, we have $\Psi_0 dz = bA^{(1,0)}$ for some function $b\colon T^2 \to \mathbb{C}$. Then

$$d^\nabla(\Psi_0 dz) = \nabla\Psi_0 \wedge dz = db \wedge A^{(1,0)} + b d^\nabla A^{(1,0)} \tag{44}$$

and from (16)

$$0 = d^\nabla * A = d^\nabla(JA) = \nabla J \wedge A + J d^\nabla A = -2JA \wedge A + J d^\nabla A,$$

where the last equality used $\nabla J = 2(*Q - *A)$ and $Q \wedge A = 0$. Thus

$$d^\nabla A^{(1,0)} = \frac{1}{2} d^\nabla(A - I * A) = A \wedge A. \tag{45}$$

Since Ψ is holomorphic,

$$\left(\nabla^{(0,1)} + (\mu^{-1} - 1)\,\mathrm{ad}_{A^{(1,0)}}\right)(\Psi_0 + \Psi_1\mu^{-1} + \cdots) = 0,$$

and equating constant terms gives

$$\nabla^{(0,1)}\Psi_0 = [A^{(0,1)}, \Psi_0]. \tag{46}$$

Substituting this and (45) into (44) gives $db \wedge A^{(1,0)} = 0$ and hence $\bar\partial b = 0$ so we conclude that b is constant. Scaling Ψ by powers of μ if necessary, we may assume that Ψ_0 is not the zero function and hence $b \neq 0$, proving the lemma. $\square$

We henceforth choose Ψ as in Lemma 4.3. Since $\Psi(p,\mu)$ is parallel, $\mathcal{E}_\mu(p)$ and $\widetilde{\mathcal{E}}_\mu(p)$ are the eigenlines of $\Psi(p,\mu)$. Using $*A = JA = -AJ$, we see

$$\operatorname{im} A^{(1,0)} \subseteq E \subseteq \ker A^{(1,0)}$$

for the $+i$-eigenspace E of J, and since $A^{(1,0)}$ is nowhere vanishing these are equalities. Thus $A^{(1,0)}(p)$ is nilpotent, with sole eigenvalue zero and just one eigenspace, namely $E(p)$. From Lemma 4.3, $A^{(1,0)}$ and Ψ_0 have common eigenspaces, and we conclude that

$$\lim_{\mu\to\infty} \mathcal{E}_\mu(p) = \lim_{\mu\to\infty} \widetilde{\mathcal{E}}_\mu(p) = E(p). \tag{47}$$

From the reality condition (19) we see also that as $\mu \to 0$, the limit of the eigenlines is $E(p)j$. This concludes the proof of Theorem 4.2 (i).

We proceed now to the proof of (iii) of Theorem 4.2, which will include the statement and proof of Lemma 4.4. Let $pr : \mathbb{CP}^1 \to \mathbb{CP}^1$ be the double cover $pr(\zeta) = \zeta^2$. The eigenlines of $\Psi(p)$ define a line bundle $\mathcal{E}$ on $T^2 \times (pr)^{-1}(U)$, and writing $\mathcal{E}^\zeta := \mathcal{E}|_{T^2\times\{\zeta\}}$, the holomorphic structure on each $\mathcal{E}^\zeta$ is given by $(\nabla^{\zeta^2})^{(0,1)}$. From (47), the bundle $\mathcal{E}$ extends holomorphically over $\zeta = \infty$. For each $\zeta \in (pr)^{-1}(U)$, the line bundle $\mathcal{E}^\zeta$ carries the flat connection ∇^{ζ^2}, so has degree 0 and hence is smoothly trivialisable. Since degree is constant in families, $\mathcal{E}^\infty = E$ also has degree 0 and we may choose a holomorphic family of smooth trivialising sections

$$\varphi(p,\zeta) = \varphi_0(p) + \zeta^{-1}\varphi_1(p) + \zeta^{-2}\varphi_2(p) + \cdots \quad \text{for } \zeta \in (pr)^{-1}(U) \cup \{\infty\}. \tag{48}$$

Denote by Ω_ζ the connection form of the restriction of ∇^{ζ^2} to $\mathcal{E}^\zeta$ with respect to this trivialisation, that is

$$(\nabla + (\zeta^2-1)A^{(1,0)} + (\zeta^{-2}-1)A^{(0,1)})(\varphi_0 + \zeta^{-1}\varphi_1 \cdots) = \Omega_\zeta(\varphi_0 + \zeta^{-1}\varphi_1 + \cdots)\,, \tag{49}$$

and $\Omega_\zeta = \zeta^2\omega_{-2} + \zeta\omega_{-1} + \omega_0 + \zeta^{-1}\omega_1 + \cdots$. Let k be a function on the universal cover $\mathbb{C}$ of T^2 so that $k(p)\varphi(p,\zeta)$ is parallel with respect to ∇^{ζ^2}. Then

$$dk = -\Omega_\zeta k\,.$$

Integrating gives

$$\log h(\zeta) = \log\left(\frac{k(p+\gamma)}{k(p)}\right) = -\int_\gamma \Omega_\zeta.$$

Thus to prove the first equation in Theorem 4.2 (iii) it suffices to show that for ζ near ∞, the connection form may be written as

$$\Omega_\zeta = \zeta w_{-1} dz + \omega_0(p) + \zeta^{-1}\omega_1(p) + \cdots \quad \text{for } w_{-1} \text{ a non-zero constant.}$$

The section φ_0 trivialises the bundle $\mathcal{E}^\infty = E$ and hence is nowhere vanishing, so since $A^{(1,0)}$ has sole eigenvalue zero the ζ^2 term of (49) gives that $\omega_{-2} = 0$. Using Lemma 4.3, the ζ term of (49) is

$$\Psi_0 \varphi_1 dz = \varphi_0 \omega_{-1}. \tag{50}$$

Using that $\Psi(p, \mu)$ is trace-free,

$$\begin{aligned} \det \Psi(p,\mu) &= -\frac{1}{2} \operatorname{tr}\left(\Psi(p,\mu)^2\right) \\ &= \det \Psi_0(p) - \operatorname{tr}(\Psi_0(p)\Psi_1(p))\mu^{-1} + c_2\mu^{-2} + \cdots \\ &= \mu^{-1}(\operatorname{tr}(\Psi_0(p)\Psi_1(p)) + c_1(p)\mu^{-1} + \cdots). \end{aligned}$$

Writing $\zeta^2 = \mu$, we see that the eigenvalues $\pm a(p,\zeta)$ of $\Psi(p,\mu)$ are of the form

$$a(p,\zeta) = a_1(p)\zeta^{-1} + a_2(p)\zeta^{-2} + \cdots, \text{ where } a_1^2(p) = -\operatorname{tr}(\Psi_0(p)\Psi_1(p)).$$

The section φ is an eigenvector of Ψ and the ζ^{-1} term of the eigenvector equation is

$$\Psi_0 \varphi_1 = a_1 \varphi_0. \tag{51}$$

Combining this with (50) gives $\omega_{-1} = a_1 dz$ so that

$$\Omega_\zeta = \zeta a_1(p) dz + \omega_0(p) + \zeta^{-1}\omega_1(p) + \cdots, \text{ where } a_1^2(p) = -\operatorname{tr}(\Psi_0(p)\Psi_1(p)).$$

The first equation in Theorem 4.2 (iii) now follows from the following lemma.

Lemma 4.4. *The trace* $\operatorname{tr}(\Psi_0\Psi_1)$ *is a non-zero constant.*

Proof. We first show that this trace vanishes if and only if $\nabla - A$ preserves $E = \ker \Psi_0$. From (46) we know that $\ker \Psi_0$ is preserved by $(\nabla - A)^{(0,1)}$. On the other hand (43) and Lemma 4.3 show that $(\nabla - \mathrm{ad}_A)^{(1,0)}\Psi_0 = [\Psi_1, \Psi_0]dz$. Thus $\nabla - A$ preserves E if and only if $[\Psi_1, \Psi_0]$ is a multiple of Ψ_0, which is equivalent to $\operatorname{tr}(\Psi_0\Psi_1) = 0$. Restricting

$$(\nabla - \mathrm{ad}_A)J = 2QJ$$

to E and recalling that Q interchanges E and Ej, we have $\operatorname{tr}(\Psi_0\Psi_1) \neq 0$. Since Ψ_0, Ψ_1 are holomorphic on T^2, $\operatorname{tr}(\Psi_0\Psi_1)$ is constant. □

From the reality condition (19) we have

$$\log h(\bar{\zeta}^{-1}) = -\log h(\zeta)\,,$$

which completes the proof of Theorem 4.2 (iii). For (ii), we observe that since a_1 and φ_0 are nowhere vanishing, from (51) the same is true of φ_1, so the eigenlines of the holonomy agree only to first-order at $\zeta = \infty$. This concludes our verification of Theorem 4.2. □

Let q be the unique (up to real scaling) quaternionic Hermitian form on V/L that is parallel with respect to ∇, and denote by

$$q = q_{\mathbb{C}} + j \det$$

its splitting into a complex valued hermitian form and a complex-linear non-degenerate 2-form with respect to multiplication by the constant quaternion i. We use det to measure the order to which eigenlines agree. Define a polynomial

$$P(\mu) = \prod (\mu - \mu_\alpha)^{n_\alpha}$$

by the condition that $\mu_\alpha \in \mathbb{C}_*$ is a zero of P of order n_α if and only if $\mathcal{E}_{\mu_\alpha}(p)$ and $\widetilde{\mathcal{E}}_{\mu_\alpha}(p)$ agree to order n_α, as measured by the order of vanishing of the 2-form det. By (19) the polynomial P is preserved by $\mu \mapsto \bar{\mu}^{-1}$. From Theorem 4.2 the eigenlines of the holonomy have a unique limit at each of $\mu = 0$ and $\mu = \infty$ and the two eigenlines agree there only to first-order. Thus we define the *eigenline spectral curve* Σ_e to be the curve

$$y^2 = \mu P(\mu)\,.$$

By construction, for each $p \in T^2$ the eigenlines of $H^\mu_\gamma(p)$ define a line bundle $\mathcal{E}(p)$ on Σ_e which by the above theorem extends holomorphically to $P_\infty = y^{-1}(\infty)$ and $P_0 = y^{-1}(0)$. We call the line bundles $\mathcal{E}(p)$ *eigenline bundles.* The restriction of these bundles to an open set without singular points is holomorphic, and since Σ_e is smooth in a neighbourhood of P_∞ and P_0, we may define the eigenline bundles over these points by holomorphic extension. The symmetry of P ensures that Σ_e possesses the fixed point free real structure ρ corresponding to the action of the quaternion j on V/L, that is, $\rho^*\mathcal{E}(p) = \mathcal{E}(p)j$. We can similarly define a polynomial $Q(\zeta)$ using the holonomy of the family ∇^ζ. Taking $\mu = \zeta^2$, since the connections ∇^μ and ∇^ζ are gauge equivalent, the two eigenlines of the holonomy of ∇^ζ agree to the same order as the eigenlines of the holonomy of ∇^μ. Thus

$$Q(\zeta) = P(\mu).$$

Define a hyperelliptic curve Y by

$$\nu^2 = Q(\zeta);$$

this is the spectral curve used in [13]. We have shown above that Q is preserved under $\zeta \mapsto -\zeta$ and we denote by $\tau(\zeta, \nu) = (-\zeta, -\nu)$ the corresponding fixed point free holomorphic involution of Y. We then obtain

Corollary 4.5. *The eigenline spectral curve is the quotient*

$$\Sigma_e \cong Y/\tau$$

of the curve Y by a holomorphic involution without fixed points.

The key property of these curves is that they have finite genus, and so we are in the realm of algebraic geometry. This was proven for Y in [13] and for Σ_e in [9, Sec. 6.3]. We note that working with a quaternionic line bundle

rather than a complex rank two vector bundle makes the finite genus result easier to prove for Σ_e than to prove directly for Y, so this corollary yields a simplification of the proof in [13].

Denote by Σ_e° the open eigenline spectral curve $\Sigma_e \setminus \{P_0, P_\infty\}$.

Theorem 4.6. *For a constant mean curvature immersion $f\colon T^2 \to \mathbb{R}^3$ of a 2–torus the space of μ-Darboux transforms $\widehat{f}\colon T^2 \to \mathbb{R}^4$ of f is given by the quotient of its open eigenline spectral curve by the (fixed point free) real structure ρ, together with finitely many complex projective lines*

$$\{\ \mu\text{-Darboux transforms } \widehat{f}\colon T^2 \to \mathbb{R}^4,\ \mu \in \mathbb{C}_*\} = \Sigma_e^\circ/\rho \cup \mathbb{CP}^1 \cup \ldots \cup \mathbb{CP}^1,$$

where the projective lines are distinct and each intersects Σ_e°/ρ in one or two points. The pair (P_0, P_∞) corresponds to the original immersion f.

Proof. For each $x \in \Sigma_e^\circ$, choose a $\nabla^{\mu(x)}$-parallel section φ^x satisfying $\varphi^x(p)\mathbb{C} = \mathcal{E}_x(p)$ for $p \in T^2$, and let $\widehat{\varphi}^x$ be the prolongation (5) of φ^x. The map $\widehat{f}^x = \widehat{\varphi}^x\mathbb{H}\colon T^2 \to S^4$ is by definition a μ-Darboux transform of f. If the holonomy H^μ_γ has distinct eigenspaces, then x clearly uniquely determines $\widehat{f}^x$. From [9] we know that there are only finitely many $x \in \Sigma_e^\circ$ such that the holonomy has a two-dimensional eigenspace and hence we may extend the map $x \mapsto \widehat{f}^x$ to such points. By (19), the section $\varphi^x j$ is parallel for $\nabla^{\mu(\rho(x))}$, so the points x and $\rho(x)$ give rise to the same μ–Darboux transform and we obtain a well–defined map

$$\begin{aligned} \Sigma_e^\circ/\rho &\to \{\mu\text{-Darboux transforms } \widehat{f}\colon T^2 \to \mathbb{R}^4 \text{ of } f\} \\ x &\mapsto \widehat{f}^x = \widehat{\varphi}^x\mathbb{H}\,. \end{aligned}$$

It is proven in [13] that ρ acts without fixed points. In Theorem 6.1 we prove that the points P_0 and P_∞ correspond to the original immersion f.

Suppose that $\widehat{f}^{x_1} = \widehat{f}^{x_2}$, i.e. $\widehat{\varphi}^{\mu_2} = \widehat{\varphi}^{\mu_1} g$ for $g : \mathbb{R}^2 \to \mathbb{H}_*$. Then

$$\pi d\widehat{\varphi}^{\mu_2} = (\pi d\widehat{\varphi}^{\mu_1})g + \pi\widehat{\varphi}^{\mu_1} dg$$

so from (5) we see that $g = v + wj$ is constant. Thus

$$\begin{aligned} 0 &= \nabla^{\mu_2}(\varphi_1 g) = (\nabla^{\mu_2}\varphi_1)v + (\nabla^{\bar{\mu}_2^{-1}}\varphi_1)wj \\ &= (\mu_2 - \mu_1)A^{(1,0)}\varphi_1 v + (\bar{\mu}_2^{-1} - \mu_1)A^{(1,0)}\varphi_1 wj + (\mu_2^{-1} - \mu_1^{-1})A^{(0,1)}\varphi_1 v \\ &\quad +(\bar{\mu}_2 - \mu_1^{-1})A^{(0,1)}\varphi_1 wj, \end{aligned}$$

where the first and last terms take values in $E = \operatorname{im} A^{(1,0)}$, and the remaining terms are valued in $Ej = \operatorname{im} A^{(0,1)}$. Since the $(1,0)$ and $(0,1)$ parts each separately vanish, each of the four terms above is zero and so x_2 is either x_1 or $\rho(x_1)$.

Suppose that $\mu \in S^1$ is such that the holonomy matrix $H^\mu_\gamma(p) = H^{\rho(\mu)}_\gamma(p)$ has a two-dimensional eigenspace. The connection ∇^μ has $\mathbf{SU}(2)$ holonomy and the limiting lines $\mathcal{E}_\mu = \lim_{\eta\to\mu}\mathcal{E}_\eta$ and $\widetilde{\mathcal{E}}_\mu = \lim_{\eta\to\mu}\widetilde{\mathcal{E}}_\eta$ satisfy $\widetilde{\mathcal{E}}_\mu = \mathcal{E}_\mu j$. In particular $\mathcal{E}_\mu$ and $\mathcal{E}_\mu j$ span the two-dimensional eigenspace and each point

in this eigenspace yields the same μ-Darboux transform, corresponding to the point in Σ_e/ρ with this μ-value.

We turn our attention then to the finitely many $\mu \in \mathbb{C}_* \setminus S^1$ for which the holonomy matrices $H^\mu_\gamma(p)$ and $H^{\rho(\mu)}_\gamma(p)$ each have two-dimensional eigenspaces. In this case, writing $\mathcal{E}_\mu$ again for the limiting eigenline, $\mathcal{E}_\mu j$ does not belong to the two-dimensional eigenspace W of H^μ. Hence the μ–Darboux transforms associated to the pair $(\mu, \rho(\mu))$ are parameterised by $\mathbb{P}(W) \simeq \mathbb{CP}^1$. If Σ_e° is branched over μ there is exactly one μ–Darboux transform given by the pair $(x, \rho(x)) \in \Sigma_e^\circ$ with $\mu(x) = \mu$, and the $\mathbb{CP}^1$ intersects Σ_e°/ρ in a single point. When Σ_e° is unbranched over $\mu \in \mathbb{C}_* \setminus S^1$, we have two limiting eigenlines $\mathcal{E}_\mu$ and $\tilde{\mathcal{E}}_\mu \neq \mathcal{E}_\mu j$ and the $\mathbb{CP}^1$ intersects Σ_e°/ρ in two points. □

5. The multiplier spectral curve

We introduce a one-dimensional analytic variety, called the *multiplier spectral curve* [4], which has the advantage that it may be defined for any conformal immersion of a 2-torus into S^4 with degree zero normal bundle. In general this variety may have infinite geometric genus, but we show that in the case of a constant mean curvature torus in $\mathbb{R}^3$ its geometric genus is finite. Indeed, its normalisation completes to a compact Riemann surface biholomorphic to the normalisation of the eigenline curve. The multiplier curve and the eigenline curve are not in general isomorphic since the multiplier curve is always singular (Corollary 5.5).

As $\pi_1(T^2)$ is abelian the holonomy of a section of $\widetilde{V/L}$ (the pullback of V/L to the universal cover $\mathbb{C}$ of T^2) lies in an abelian subgroup of $\mathbb{H}_*$. We assume that we have conjugated so that this subgroup is equal to $\mathbb{C}_*$.

Definition 5.1. The *multiplier spectral curve* Σ_m of a conformal immersion $f : T^2 \to S^4 \simeq \mathbb{HP}^1$ is the set of holonomies realised by holomorphic sections of $\widetilde{V/L}$. Denote by $H^0_h(\widetilde{V/L})$ the space of holomorphic sections φ with multiplier $h \in \mathrm{Hom}(\pi_1(T^2), \mathbb{C}_*)$, that is $\gamma^*\varphi = \varphi h_\gamma$ for all $\gamma \in \pi_1(T^2)$. Then

$$\Sigma_m \quad = \quad \{h \in \mathrm{Hom}(\pi_1(T^2), \mathbb{C}_*) |\text{ there exists } 0 \not\equiv \varphi \in H^0_h(\widetilde{V/L})\}\,.$$

Let $\mathrm{Harm}(T^2, \mathbb{C})$ be the space of harmonic 1-forms on T^2, then there is a natural isomorphism

$$\begin{aligned} \mathrm{Harm}(T^2, \mathbb{C})/\Gamma^* \quad &\to \quad \mathrm{Hom}(\pi_1(T^2), \mathbb{C}_*) \\ \omega \quad &\mapsto \quad \exp \int \omega \end{aligned}$$

where Γ^* denotes the harmonic 1-forms with periods in $2\pi i\mathbb{Z}$. The lift $\log \Sigma_m$ of the multiplier spectral curve to $\mathrm{Harm}(T^2, \mathbb{C})$ is the space on which a holomorphic family of elliptic operators has nontrivial kernel [4, 5]. One concludes that $\log \Sigma_m$ and hence $\Sigma_m = \log \Sigma_m/\Gamma^*$ are one-dimensional analytic varieties, justifying our terminology. Notice that these curves possess a real structure σ given by complex conjugation of the holonomy.

The multiplier curve allows us to give a geometric description of the space of Darboux transforms.

Theorem 5.2. *For an immersed torus $f : T^2 \to \mathbb{R}^3$ of constant mean curvature the set of all Darboux transforms of f is parameterised by the quotient of the multiplier spectral curve under the real structure σ together with at most countably many complex and quaternionic projective spaces:*

$$\{\textit{Darboux transforms}\} = \Sigma_m/\sigma \cup \bigcup_{m=1}^{\infty} \mathbb{CP}^{k_m} \cup \bigcup_{n=1}^{\infty} \mathbb{HP}^{l_n}.$$

Proof. A section of $\widetilde{V/L}$ is holomorphic when it lies in the kernel of the quaternionic holomorphic structure D (introduced in (4)) with multiplier h. For all but a discrete set of $h \in \Sigma_m$, the space $H^0_h(\widetilde{V/L})$ is only complex one-dimensional [4, Theorem 3.3]. The Darboux transform given by a holomorphic section φ of $\widetilde{V/L}$ is unchanged by quaternionic scaling of this section, and if φ has complex multiplier h, then φj has multiplier $\bar{h} = \sigma(h)$. Thus away from a discrete set, to each pair $(h, \sigma(h)) \in \Sigma_m/\sigma$ there corresponds a unique Darboux transform.

If $H^0_h(\widetilde{V/L})$ has complex dimension $k+1$, then the same is true also for $\sigma(h) = \bar{h}$ since multiplying by j interchanges the two spaces of sections. Thus if $h \notin \mathbb{R}$, the space of Darboux transforms with multiplier h or $\sigma(h)$ is parameterised by $\mathbb{CP}^k$. If $h \in \mathbb{R}$ the space $H^0_h(\widetilde{V/L})$ is a quaternionic vector space and thus the set of corresponding Darboux transforms is parameterised by $\mathbb{HP}^l$ with $l = \frac{k-1}{2}$. □

The eigenvalues of $H^\mu_\gamma(p)$ give a well-defined holomorphic function $h_\gamma(x)$ on Σ_e°, and we write

$$\begin{array}{rccl} h\colon & \Sigma_e^\circ & \to & \Sigma_m \\ & x & \mapsto & (h\colon \gamma \mapsto h_\gamma(x)) \,. \end{array}$$

Let $\widetilde{\Sigma}_e$ denote the normalisation of the eigenline spectral curve Σ_e and note as before that any singularities of Σ_e are contained in Σ_e°. We write $\widetilde{\Sigma}_m$ and $\log \widetilde{\Sigma}_m$ for the (analytic) normalisations of the multiplier spectral curve Σ_m and of $\log \Sigma_m$, and observe that $\log \widetilde{\Sigma}_m/\Gamma^* = \widetilde{\Sigma}_m$. Let

$$\tilde{h} : \widetilde{\Sigma}_e^\circ \to \widetilde{\Sigma}_m$$

be the lifting of h to the normalisations.

Theorem 5.3. *The multiplier curve of a constant mean curvature torus is connected, and its normalisation can be completed to a compact Riemann surface biholomorphic to the normalisation of the eigenline spectral curve. This biholomorphism is given by (an extension of) the map $\tilde{h}$ defined above.*

Proof. We show first that $\tilde{h}$ is injective. It suffices to prove this away from a discrete set, as $\tilde{h}$ is a holomorphic map between two Riemann surfaces. Write $\pi_e : \widetilde{\Sigma}_e \to \Sigma_e$ for the normalisation map. We show that $\tilde{h}$ is injective on

$$U = \{x \in \widetilde{\Sigma}_e^\circ \colon \dim H^0_{h(\pi_e(x))}(\widetilde{V/L}) = 1\} \setminus \pi_e^{-1}(S_e \cup h^{-1}(S_m)),$$

where S_e, S_m are the singular points of the two spectral curves. Since we are omitting these points, we do not need to distinguish between h and $\tilde{h}$. As mentioned before, the set $\{h \in \Sigma_m \colon \dim H^0_h(\widetilde{V/L}) > 1\}$ is discrete [4]. Its pre-image under the holomorphic map $\tilde{h}$ is thus also discrete, and so U is the complement of a discrete set.

Each $x \in U$ determines a unique $\nabla^{\mu(x)}$–parallel section $\varphi^x \in H^0_{h(x)}(V/L)$ up to complex scaling. If

$$h(x) = h(x^\#) \tag{52}$$

then $\varphi^x = \varphi^{x^\#}\lambda$ with $\lambda \in \mathbb{C}_*$ since $\dim H^0_{h(x)}(\widetilde{V/L}) = 1$. In particular φ^x is parallel with respect to both $\nabla^{\mu(x)}$ and $\nabla^{\mu(x^\#)}$. Therefore (18) and the fact that $\ker A^{(1,0)} \cap \ker A^{(0,1)} = \{0\}$ imply that $\mu(x^\#) = \mu(x)$. If x and $x^\#$ are exchanged by the hyperelliptic involution then by (52) we see that $h(x) = h(x^\#) = \pm 1$. Since $\dim H^0_{h(x)}(\widetilde{V/L}) = 1$ we conclude that $x = x^\#$ is a branch point of μ and so $\tilde{h}$ is injective.

We now show that the map $\tilde{h}$ holomorphically extends to $P_0 = \mu^{-1}(0)$ after completing one end of the multiplier spectral curve by a single point. It suffices to extend $\tilde{h}_\gamma$ for an appropriate $\gamma \in \Gamma$. In this proof we will not notationally distinguish between h_γ and h. The eigenline spectral curve Σ_e is branched at P_0 hence ζ with $\zeta^2 = \mu$ is a local coordinate near P_0. Theorem 4.2 shows that $\log h(\zeta)$ is a chart around the end of the multiplier spectral curve which is completed by adding the point 0 in this chart. An analogous argument can be used at P_∞ and thus the normalised multiplier spectral curve becomes a compact Riemann surface $\overline{\widetilde{\Sigma}}_m$ by adding these two points. Moreover, $\tilde{h} \colon \widetilde{\Sigma}_e \to \overline{\widetilde{\Sigma}}_m$ extends to an injective holomorphic map. For a general conformal branched immersion $T^2 \to S^4$, the multiplier spectral curve Σ_m consists either of two components each of which has one end, or of a single component with two ends [4]. Since both ends of $\widetilde{\Sigma}_m$ are contained in the image of $\tilde{h}$ the multiplier spectral curve of a constant mean curvature torus is connected and $\tilde{h}$ is a biholomorphism. □

The space $H^0_h(\widetilde{V/L})$ of holomorphic sections with multiplier h is generically complex one-dimensional. It defines a line bundle $\mathcal{L} \to \widetilde{\Sigma}_m$, the *kernel bundle*, on the normalisation of the multiplier spectral curve [4, Theorem 3.6] with fibres $\mathcal{L}_{\tilde{h}} = \varphi\mathbb{C}$ where $\varphi \in H^0_h(\widetilde{V/L})$ and h denotes the image of $\tilde{h} \in \widetilde{\Sigma}_m$ under the normalisation map $\pi_m \colon \widetilde{\Sigma}_m \to \Sigma_m$. By Theorem 5.3 each

point in $\widetilde{\Sigma}_m$ is $\tilde{h}(x)$ for a $x \in \widetilde{\Sigma}_e$, where $\tilde{h}$ denotes the biholomorphism of the Theorem. Hence φ is parallel with respect to $\nabla^{\mu(x)}$ and therefore has no zeros. This enables us to define for each $p \in T^2$ a holomorphic line bundle $\mathcal{L}(p)$ where $\mathcal{L}_{\tilde{h}}(p) = \varphi(p)\mathbb{C}$.

Corollary 5.4. *Let $f\colon T^2 \to \mathbb{R}^3$ be a constant mean curvature torus. For each $p \in T^2$ the push forward of the eigenline bundle $\mathcal{E}(p)$ under the biholomorphism $\tilde{h}$ is the evaluation $\mathcal{L}(p)$ of the kernel bundle*

$$\tilde{h}_* \mathcal{E}(p) = \mathcal{L}(p)\,.$$

Consequently, every Darboux transform $\widehat{f}$ of f given by a holomorphic section in $\mathcal{L}_{\tilde{h}}$ with $\tilde{h} \in \widetilde{\Sigma}_m$ is a μ-Darboux transform of f.

As we have seen the normalisations of the eigenline and the multiplier spectral curve are biholomorphic. However the two spectral curves are not in general isomorphic.

Corollary 5.5. *The multiplier spectral curve of a constant mean curvature torus is always singular. The eigenline spectral curve is a partial desingularisation of the multiplier spectral curve.*

Proof. The projection of Σ_e to the μ-plane has at least one pair $x, \rho(x)$ of ramification points over some $\mu, \bar{\mu}^{-1} \in \mathbb{C}_* \setminus S^1$. At these points, for each generator $\gamma \in \pi_1(T^2, p)$ the eigenvalues of the holonomy matrix $H^\mu_\gamma(p) \in SL(2,\mathbb{C})$ are either both 1 or -1. Thus the representation $h(x)\colon \pi_1(T^2) \to \mathbb{C}_*$ is in particular real and so $h(\rho(x)) = \overline{h(x)} = h(x)$, showing that the map $h\colon \Sigma_e^\circ \to \Sigma_m$ is not one-to-one. Since Σ_e is smooth at P_0 and P_∞ the normalisation maps give an extension $\pi_m \circ \pi_e^{-1}$ of h to all of Σ_e . We then have that the normalisation map for Σ_m factors as $\pi_m = h \circ \pi_e$ and that it is not one-to-one, proving our claims. We note that if we replace the original torus by a four-fold cover then all ramification points other than P_0, P_∞ map to the constant multiplier 1. ☐

6. Geometric Picture

The spectral curves encode geometric information about the original constant mean curvature torus $f : T^2 \to \mathbb{R}^3$. In this section we show that f is the limit of μ–Darboux transforms as μ tends to 0 or ∞.

The eigenline bundle $\mathcal{E}(p) \to \Sigma_e$ based at $p \in T^2$ gives rise to a complex line bundle $\mathcal{E} \to T^2 \times \Sigma_e^\circ$ via parallel transport of the fiber $\mathcal{E}(p)_x$ over $x \in \Sigma_e^\circ$ by the connection $\nabla^{\mu(x)}$. Since $\nabla^{\mu(x)}$ has a simple pole at P_0 and at P_∞, parallel transport is only defined on $\Sigma_e^\circ = \Sigma_e \setminus \{P_0, P_\infty\}$.

Then $\mathcal{E}_x = \varphi^x \mathbb{C}$ where φ^x is a $\nabla^{\mu(x)}$–parallel section with multiplier $h(x)$. The prolongation $\widehat{\varphi}^x$ of φ^x (see Lemma 2.1) defines a complex line subbundle

$$\widehat{\mathcal{E}} \to T^2 \times \Sigma_e^\circ$$

of the trivial $\mathbb{H}^2$ bundle V which is holomorphic over Σ_e°. The Darboux transform for $x \in \Sigma_e^\circ$ is the map $\widehat{f}^x = \widehat{\mathcal{E}}_x\mathbb{H}\colon T^2 \to S^4$.

Theorem 6.1. *Let $f : T^2 \to \mathbb{R}^3$ be a constant mean curvature torus with corresponding quaternionic line subbundle $L \subset V$ and $\delta\colon L \to KV/L$ be the derivative of f as defined in* (1)*. For every $p \in T^2$ the line bundle $\widehat{\mathcal{E}}(p) \to \Sigma_e^\circ$ extends holomorphically across P_0 and P_∞ with*

$$\lim_{x\to P_0} \widehat{\mathcal{E}}_x = \delta^{-1}(Ej) \quad \text{and} \quad \lim_{x\to P_\infty} \widehat{\mathcal{E}}_x = \delta^{-1}(E)\,.$$

Hence, when x tends to P_0 or P_∞, the Darboux transforms $\widehat{f}^x$ limit to the original constant mean curvature torus f.

Proof. As f is an immersion δ has no zeros, and from (21) and (18) we see that $\widehat{\varphi}^x = \varphi^x - \delta^{-1}(\alpha^\mu\varphi^x)$, where

$$\alpha^\mu = \nabla^\mu - \nabla = \mu A^{(1,0)} + \mu^{-1}A^{(0,1)} - A\,.$$

From (48) we have

$$\varphi^x = \varphi_0 + \varphi_1\zeta^{-1} + \dots$$

with $\varphi^x\mathbb{C} = \mathcal{E}_x$. From Theorem 4.2 we know that φ_0 is a nowhere vanishing section of $E = \ker A^{(1,0)}$. Then

$$\begin{aligned}\widehat{\varphi}^x &= (\varphi_0 + \varphi_1\zeta^{-1} + \dots - \delta^{-1}(\zeta^2A^{(1,0)} + \zeta^{-2}A^{(0,1)} - A)(\varphi_0 + \varphi_1\zeta^{-1} + \cdots))\\ &= -\zeta\delta^{-1}(A^{(1,0)}\varphi_1) + \text{ lower order terms}\end{aligned}$$

so using Lemma 4.3, Lemma 4.4 and (51) we have $A^{(1,0)}\varphi_1 \neq 0$. Since $\operatorname{im} A^{(1,0)} = E$ we conclude that $\lim_{x\to P_\infty}\widehat{\mathcal{E}}_x = \delta^{-1}(E)$. Furthermore, the limit of $\widehat{f}^x = \widehat{\mathcal{E}}_x\mathbb{H}$ as x tends to P_0 or P_∞ is $\delta^{-1}(E \oplus Ej) = L$. $\square$

Historically, points on the eigenline curve could be interpreted differential geometrically only for unitary μ, where they correspond to the associated family of constant mean curvature surfaces. From our results we obtain an interpretation of all points on the spectral curve of a constant mean curvature torus as its μ–Darboux transforms. We combine those observations in the following

Theorem 6.2. *Let $f : T^2 \to \mathbb{R}^3$ be a constant mean curvature immersion and $\pi\colon \mathbb{CP}^3 \to \mathbb{HP}^1$ the twistor projection which sends a complex line in $\mathbb{C}^4$ to the corresponding quaternionic line in $\mathbb{H}^2$. The maps*

$$\begin{tikzcd} & \mathbb{CP}^3 \arrow[d, "\pi"] \\ T^2 \times \Sigma_e \arrow[ur, "\widehat{\mathcal{E}}"] \arrow[r, "\widehat{\mathcal{E}}\mathbb{H}"'] & \mathbb{HP}^1 \end{tikzcd}$$

defined by the prolongation of the eigenline bundles together with the twistor projection satisfy:

(i) *For each $x \in \Sigma_e^\circ$, $\widehat{f}^x = \pi\widehat{\mathcal{E}}(\cdot, x)$ is a μ-Darboux transform of the original constant mean curvature immersion f. In fact, $\widehat{f}^x$ is also a constant mean curvature torus in Euclidean 3–space (Theorem 3.6).*

(ii) *The original constant mean curvature torus $f = \pi\widehat{\mathcal{E}}(\cdot, P_\infty) = \pi\widehat{\mathcal{E}}(\cdot, P_0)$ is the limit of μ–Darboux transforms for $\mu \to 0, \infty$.*

(iii) *The Gauss map of f is given by $\mathcal{E}(\cdot, P_\infty) = E$ as the $+i$ eigenspace of J.*

(iv) *For $p \in T^2$ the eigenline curve is algebraically mapped into $\mathbb{CP}^3$ by $\widehat{\mathcal{E}}(p, \cdot)$. Thus we obtain a smooth T^2–family of algebraic curves $\Sigma_e \to \mathbb{CP}^3$.*

By pulling back the eigenline bundles and their prolongations, the analogous result holds on the normalisation $\widetilde{\Sigma}_e \cong \widetilde{\Sigma}_m$. This normalised version holds more generally for conformally immersed tori into S^4 of finite spectral genus [4], for which there is a multiplier curve but no eigenline curve. The proof of this result is much more involved and requires asymptotic analysis of Dirac–type operators [5]. For constant mean curvature tori our proof can be seen as a geometric interpretation of the eigenline spectral curve [13]. In fact, the harmonic Gauss map of the constant mean curvature torus is described by the T^2–family of algebraic functions $\mathcal{E}\colon T^2 \times \Sigma_e \to \mathbb{CP}^1$ which, interpreted as a flow of line bundles, is linear in the Jacobian of Σ_e. From this one could show that the prolongation bundle $\widehat{\mathcal{E}}$ also gives a linear T^2–flow in the Jacobian. In the generic case when Σ_e is smooth, such flows can be parametrised by theta functions of Σ_e, providing explicit conformal parametrisations of the constant mean curvature torus and all its spectral μ–Darboux transforms [2].

Appendix A. Darboux Transforms of the Standard Cylinder

We now illustrate some of our results by explicitly computing μ–Darboux transforms of the standard cylinder

$$f(x, y) = \frac{1}{2}(-ix + je^{iy}) \tag{53}$$

in $\mathbb{R}^3$. The derivative of f is

$$df = \frac{1}{2}(-idx + jie^{iy}dy)$$

and the Gauss map of f is given by

$$N(x, y) = -je^{iy}\,.$$

To compute μ-Darboux transforms of f, we need to find parallel sections $\varphi \in \Gamma(\widetilde{V/L})$ of the flat connection $\nabla^\mu = \nabla + *A(J(a-1)+b)$ on V/L, where a and b are defined in terms of μ (Remark 3.2). As in section 3 we identify V/L with $\underline{e\mathbb{H}}$ via the splitting $V = L \oplus \underline{e\mathbb{H}}$ where $e = \begin{pmatrix}1\\0\end{pmatrix}$. Thus, writing

$\varphi = e\alpha$ we seek $\alpha : \tilde{M} = \mathbb{R}^2 \to \mathbb{H}$ with (27)

$$d\alpha = -\frac{1}{2} df(N\alpha(a-1) + \alpha b) .$$

Using the decomposition $\alpha = \alpha_0 + j\alpha_1$ where $\alpha_0, \alpha_1 : \mathbb{R}^2 \to \mathbb{C} = \operatorname{span}\{1, i\}$, we rewrite the previous equation as

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}_x = \frac{i}{4} \begin{pmatrix} b & e^{-iy}(a-1) \\ e^{iy}(a-1) & -b \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}_y = \frac{i}{4} \begin{pmatrix} a-1 & -e^{-iy}b \\ -e^{iy}b & 1-a \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} .$$

The first differential equation has constant coefficients in x and therefore we can solve this system explicitly. Choosing a square root $c \in \mathbb{C}$ of $a-1$, the general solution of this system on $\mathbb{R}^2$ is given by

$$\alpha = \begin{pmatrix} e^{\frac{-iy}{2}} \\ p_+ e^{\frac{iy}{2}} \end{pmatrix} m_+ e^w + \begin{pmatrix} e^{\frac{-iy}{2}} \\ p_- e^{\frac{iy}{2}} \end{pmatrix} m_- e^{-w}$$

where $m_\pm \in \mathbb{C}$ and

$$w = \frac{\sqrt{2}}{4c}((a-1)x - by), \quad p_\pm = -\frac{b \pm \sqrt{2}ic}{a-1} . \tag{54}$$

Substituting the above formulae for α into (37) yields all μ–Darboux transforms $\widehat{f} = f + T$ of the standard cylinder. In general, these $\widehat{f}$ do not satisfy any periodicity conditions despite the fact that the above differential equations have periodic coefficients.

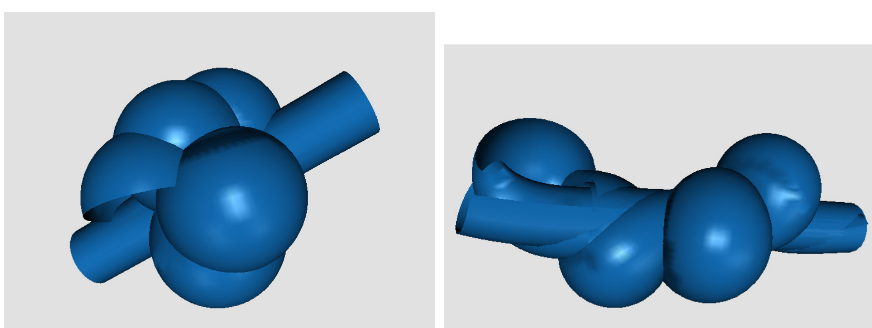

FIGURE 3. μ–Darboux transforms which are not classical Darboux transforms.

We now compute the closed μ–Darboux transforms of the standard cylinder f which are those $\widehat{f}$ satisfying

$$\widehat{f}(x, y) = \widehat{f}(x, y + 2\pi) .$$

These are given by parallel sections $\varphi \in \Gamma(\widetilde{V/L})$ of ∇^μ such that

$$\varphi(x, y + 2\pi) = \varphi(x, y)h \quad \text{with} \quad h \in \mathbb{H}_* \,. \tag{55}$$

For μ on the unit circle ∇^μ is a quaternionic connection on a rank one bundle. Then all parallel sections satisfy (55) and hence give rise to closed μ-Darboux transforms. In particular, as in Theorem 3.6, when $\mu \in S^1 \setminus \{1\}$ we obtain a translated cylinder $g = f + N + \frac{b}{1-a}$.

For μ not on the unit circle ∇^μ is not quaternionic but rather an $\mathbf{SL}(2,\mathbb{C})$-connection on the complex rank two bundle $\widetilde{V/L}$. Therefore any parallel section satisfying (55) must in fact have monodromy $h \in \mathbb{C}_*$ and is thus given by the eigenvectors of the monodromy representation of ∇^μ. For $\mu \in \mathbb{C}_*$ these parallel sections are

$$\alpha^\pm = \begin{pmatrix} e^{\frac{-iy}{2}} \\ p_\pm e^{\frac{iy}{2}} \end{pmatrix} e^{\pm w}$$

and have monodromy

$$h^\mu_\pm = -e^{\mp \frac{\sqrt{2} b \pi}{2c}} \in \mathbb{C}_* \,, \tag{56}$$

where we recall $p_\pm$ and w from (54). Therefore, the prolongations of the sections $\varphi^\pm = e\alpha^\pm$ give closed μ–Darboux transforms $\widehat{f}^\pm$. Using (37) the μ–Darboux transforms $\widehat{f}^\pm$ are rotations and translations of the original cylinder. Explicitly,

$$\widehat{f}^\pm = f + T^\pm = f + T_0^\pm + j e^{iy} T_1^\pm$$

where

$$T_0^\pm = \frac{2}{r^\pm}\left(\operatorname{Re} b - 2\frac{\bar{p}_\pm^2 i}{1 + |p_\pm|^2}\operatorname{Im} a - \frac{1 - |p_\pm|^2}{1 + |p_\pm|^2} i \operatorname{Im} b\right) \in \mathbb{C}$$

and

$$T_1^\pm = \frac{2}{r^\pm}\left(\operatorname{Re} a - 1 + \frac{1 - |p_\pm|^2}{1 + |p_\pm|^2} i \operatorname{Im} a - \frac{2p_\pm}{1 + |p_\pm|^2} i \operatorname{Im} b)\right) \in \mathbb{C}$$

with

$$r^\pm = |a - 1|^2 + |b|^2 - 4\operatorname{Im}\left((a - 1)\bar{b}\right)\frac{\operatorname{Im} p_\pm}{1 + |p_\pm|^2} \in \mathbb{R} \,.$$

We call $\mu \in \mathbb{C}_*$ a *resonance point* of f if the monodromies $h^\mu_\pm$ coincide. These are the points

$$\mu_k = 2k^2 - 1 - 2k\sqrt{k^2 - 1} \,, \qquad k \in \mathbb{Z} \setminus \{0\} \,.$$

Away from resonance points the monodromy of ∇^μ is diagonalisable with two distinct eigenvalues h^μ_+ and h^μ_-. Hence we obtain exactly two μ–Darboux transforms given by parallel sections of ∇^μ.

At the resonance points $\mu = \mu_k$ with $|k| > 1$, by (56), the monodromy is

$$h^\mu := h^\mu_+ = h^\mu_- = -1 \,.$$

The space of parallel sections of ∇^μ with monodromy $h = -1$ has dimension two and gives rise to a $\mathbb{CP}^1$– family of closed μ–Darboux transforms. A parallel section $\alpha = \alpha_+ m_+ + \alpha_- m_-$ for $m_+ = 0$ or $m_- = 0$ gives a rigid motion of the standard cylinder. However for $m_+ m_- \neq 0$ the parallel section α yields a bubbleton with $|k|$ lobes where the parameter $\frac{m_\pm}{m_-} \in \mathbb{CP}^1$ rotates and slides the bubble [24, 12].

Theorem A.1. *Every non–constant closed μ–Darboux transform $\widehat{f} : M \to \mathbb{R}^4$ of the standard cylinder f is a rigid motion of f provided μ is not a resonance point μ_k. At a resonance point μ_k we additionally obtain a $\mathbb{CP}^1$–family of bubbletons with $|k|$ lobes.*

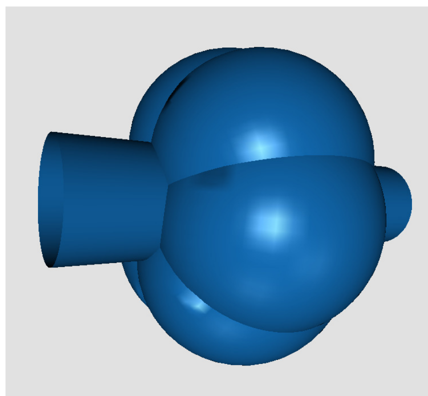

FIGURE 4. Darboux transform at the resonance point μ_6 with 6 lobes.

We now restrict our attention to those closed μ–Darboux transforms of the standard cylinder which are also classical Darboux transforms. By Theorem 3.9 these are precisely the μ–Darboux transforms for which $\mu \in \mathbb{R}_* \cup S^1$.

Corollary A.2. *The closed μ–Darboux transforms of the standard cylinder $f : M \to \mathbb{R}^3$ which are also classical are:*

(i) *the dual surface $g = f + N$ of f for $\mu \in S^1 \setminus \{1\}$, up to a translation of $\mathbb{R}^3$ in the ambient $\mathbb{R}^4$;*
(ii) *a translation of f in $\mathbb{R}^3$ along the i–axis by $\pm\frac{\sqrt{2}i}{\sqrt{a-1}}$ for $\mu \in (0, \infty)$ not a resonance point;*
(iii) *a rotation of f in $\mathbb{R}^3$ along the i-axis for $\mu \in (-\infty, 0)$ not a resonance point;*
(iv) *a bubbleton with $|k|$ lobes for a resonance point $\mu = \mu_k$ with $|k| > 1$;*
(v) *a constant point for $\mu = 1$.*

Proof. A straightforward computation, using $a^2 + b^2 = 1$, shows that

$$1 + |p_\pm|^2 = \begin{cases} \frac{2b}{a-1} p_\pm & a > 1 \\ 2 & a < 1 \end{cases}, \quad 1 - |p_\pm|^2 = \begin{cases} \pm\frac{2\sqrt{2}i}{c} p_\pm & a > 1 \\ 0 & a < 1\,. \end{cases}$$

Thus the translational part of the Darboux transform $\widehat{f} = f + T_0^{\pm} + je^{iy}T_1^{\pm}$, away from the resonance points, is given by

$$T_0^{\pm} = \begin{cases} \mp\frac{\sqrt{2}i}{c} & a > 1 \\ 0 & a < 1 \end{cases}$$

and the rotational part by

$$T_1^{\pm} = \begin{cases} 0 & a > 1 \\ \frac{2}{a-1} \mp \frac{\sqrt{2}cib}{(a-1)^2} & a < 1\,. \end{cases}$$

□

Since for each $k \in \mathbb{Z}$ a parallel section of ∇^{μ_k} gives a holomorphic section with monodromy $h = -1$, we can also take linear combinations of parallel sections $\alpha^{\pm}_{\mu_k}$ and $\alpha^{\pm}_{\mu_l}$ of ∇^{μ_k} and ∇^{μ_l} respectively with $k \neq l$. This gives a holomorphic section

$$\alpha = m_+\alpha^+_{\mu_k} + m_-\alpha^-_{\mu_k} + n_+\alpha^+_{\mu_l} + n_-\alpha^-_{\mu_l} \in H^0(\widetilde{V/L})$$

with monodromy $h = -1$ for each choice of $m_\pm, n_\pm \in \mathbb{C}$. The prolongation of α gives a closed Darboux transform of f, however α is in general not a parallel section for any $\mu \in \mathbb{C}_*$. In particular

$$\{\text{closed } \mu\text{–Darboux transforms}\} \subsetneq \{\text{closed Darboux transforms}\}\,.$$

For a further discussion of the geometry of these examples see [17].

Remark A.3. *The Darboux transformation satisfies Bianchi permutability* [4]*. In particular, a μ–Darboux transform of a bubbleton is given algebraically* [16] *and we obtain multibubbletons at the resonance points. Away from the resonance points the μ–Darboux transformation is again given by a rigid motion.*

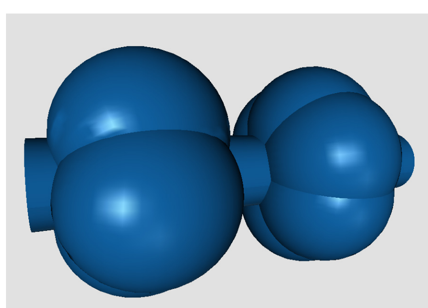

FIGURE 5. Darboux transform of a bubbleton ($\mu = \mu_3$) at the resonance point μ_6 "adding" 6 lobes.

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